

ON CLASSIFICATION OF \mathbb{Q} -FANO 3-FOLDS WITH GORENSTEIN INDEX 2 AND FANO INDEX $\frac{1}{2}$

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ABSTRACT. We generalize the theory developed by K. Takeuchi in [T1] and restrict the birational type of a \mathbb{Q} -factorial \mathbb{Q} -Fano 3-fold X with the following properties:

- (1) the Picard number of X is 1;
- (2) the Gorenstein index of X is 2;
- (3) the Fano index of X is $\frac{1}{2}$;
- (4) $h^0(-K_X) \geq 4$;
- (5) there exists an index 2 point P such that

$$(X, P) \simeq (\{xy + f(z^2, u) = 0\} / \mathbb{Z}_2(1, 1, 1, 0), o)$$

with $\text{ord} f(Z, U) = 1$.

This gives an effective bound of the value of $(-K_X)^3$ and $h^0(-K_X)$ for a \mathbb{Q} -factorial \mathbb{Q} -Fano 3-fold X with (1)~(4) by a deformation theoretic result of T. Minagawa in [Mi2].

Furthermore based on the main result, we prove that if X is a \mathbb{Q} -factorial \mathbb{Q} -Fano 3-fold with (1)~(4) and with only $\frac{1}{2}(1, 1, 1)$ -singularities as its non Gorenstein points, then

- (1) $|-K_X|$ has a member with only canonical singularities;
- (2) for any $\frac{1}{2}(1, 1, 1)$ -singularity, there is a smooth rational curve l through it such that $-K_X.l = \frac{1}{2}$;
- (3) by a blow up at a $\frac{1}{2}(1, 1, 1)$ -singularity, a flopping contraction and a smoothing of Gorenstein singularities, X can be transformed to a \mathbb{Q} -Fano 3-fold with (1)~(4) and with only $\frac{1}{2}(1, 1, 1)$ -singularities as its singularities;
- (4) X can be embedded into a weighted projective space $\mathbb{P}(1^h, 2^N)$, where $h := h^0(-K_X)$ and N is the number of $\frac{1}{2}(1, 1, 1)$ -singularities on X .

0. INTRODUCTION

We start by the definition of \mathbb{Q} -Fano variety.

Definition 0.0 (\mathbb{Q} -Fano variety). Let X be a normal projective variety. We say that X is a \mathbb{Q} -Fano variety (resp. weak \mathbb{Q} -Fano variety) if X has only terminal singularities and $-K_X$ is ample (resp. nef and big).

Let $I(X) := \min\{I | IK_X \text{ is a Cartier divisor}\}$ and we call $I(X)$ the Gorenstein index of X .

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Write $I(X)(-K_X) \equiv r(X)H(X)$, where $H(X)$ is a primitive Cartier divisor and $r(X) \in \mathbb{N}$. (Note that $H(X)$ is unique since $\text{Pic}X$ is torsion free.) Then we call $\frac{r(X)}{I(X)}$ the Fano index of X and denote it by $F(X)$.

Remark.

- (1) We can allow that a \mathbb{Q} -Fano variety (resp. a weak \mathbb{Q} -Fano variety) has worse singularities than terminal. When we have to treat such a variety in this paper, we indicate singularities which we allow, e.g., 'a \mathbb{Q} -Fano 3-fold with only canonical singularities';
- (2) if X is Gorenstein in Definition 0.0, we say that X is a Fano variety (resp. a weak Fano variety).

As an output of the minimal model program, a \mathbb{Q} -factorial \mathbb{Q} -Fano variety with Picard number 1 is an important class. We are interested in the classification of \mathbb{Q} -factorial \mathbb{Q} -Fano 3-folds with Picard number 1. Here we mention the known result about the classification of \mathbb{Q} -Fano 3-folds. G. Fano started the classification of smooth Fano 3-folds and it was completed by V. A. Iskovskih, V. V. Shokurov, T. Fujita, S. Mori and S. Mukai. S. Mukai considered Gorenstein Fano 3-folds with canonical singularities and classified them under mild assumptions. In non Gorenstein case, if Fano index is greater than 1, then the classification was obtained by T. Sano [San1] and independently by F. Campana and H. Flenner [CF] and if Fano index is 1 and only cyclic quotient terminal singularity is allowed, the classification was obtained by T. Sano [San2] (recently T. Minagawa [Mil] proved that any non Gorenstein \mathbb{Q} -Fano 3-fold with Fano index 1 can be deformed to one with only cyclic quotient terminal singularities). But if Fano index is less than 1, the only systematic result is the classification of \mathbb{Q} -Fano 3-folds which are weighted complete intersections of codimension 1 or 2 by A. R. Fletcher [Fl].

In [T1], Kiyohiko Takeuchi developed a theory to give a simple way of restricting birational type of a Fano 3-fold X with $\rho(X) = 1$ and $F(X) = 1$ and derived the bound of the genus of X and the existence of lines. In this paper, we formulate a generalization of Takeuchi's theory for a \mathbb{Q} -factorial \mathbb{Q} -Fano 3-fold X with $\rho(X) = 1$. We expect that it is useful for the classification of \mathbb{Q} -factorial \mathbb{Q} -Fano 3-folds with $\rho(X) = 1$ and $F(X) < 1$. As a test case we show that it works well under the additional assumptions that $I(X) = 2$, $F(X) = \frac{1}{2}$, $(-K_X)^3 \geq 1$ and $h^0(-K_X) \geq 1$.

Here we explain a generalization of Takeuchi's theory. Let X be a \mathbb{Q} -factorial \mathbb{Q} -Fano 3-fold with $\rho(X) = 1$. First we seek a birational morphism $f : Y \rightarrow X$ with the following properties:

- (1) Y is a weak \mathbb{Q} -Fano 3-fold;
- (2) f is an extremal divisorial contraction such that f -exceptional divisor is a prime \mathbb{Q} -Cartier divisor.

Fix a $f : Y \rightarrow X$ as above and let E be the exceptional divisor of f . Then we obtain the following diagram:

$$\begin{array}{ccc} Y & \dashrightarrow & Y' \\ f \swarrow & & \searrow f' \\ X & & X' \end{array},$$

where $Y \dashrightarrow Y'$ is an isomorphism or a composition of possibly one flop and flips and f' is a non small contraction. If $Y \dashrightarrow Y'$ is not an isomorphism, let

$$Y \xrightarrow{g_0} Y' \xrightarrow{g_1} Y'' \xrightarrow{g_2} \cdots \xrightarrow{g_{k-1}} Y^{(k)} \xrightarrow{g_k} Y'$$

be the decomposition of $Y \dashrightarrow Y'$ into flops and flips. We will see that if there is a flop while $Y \dashrightarrow Y'$, then it is g_0 . Let E_i be the strict transform of E on Y_i and \tilde{E} the strict transform of E on Y' . Note the following:

- (1) The values of $(-K_Y)^2 E$, $(-K_Y)E^2$ and E^3 are given. We know the value of $(-K_{Y_i})^3$, $(-K_{Y_i})^2 E_i$, $(-K_{Y_i})E_i^2$ and E_i^3 are decreased by f , (possibly one) flop and flips and we can express how they are decreased with some unknown quantities associated to flop and flips and so do we the value $(-K_{Y'})^3$, $(-K_{Y'})^2 \tilde{E}$, $(-K_{Y'})\tilde{E}^2$ and \tilde{E}^3 with such quantities and $(-K_X)^3$.
- (2) On the other hand the value or the relation of the value (expressed by z and u below) of $(-K_{Y'})^3$, $(-K_{Y'})^2 \tilde{E}$, $(-K_{Y'})\tilde{E}^2$ and \tilde{E}^3 are restricted by the properties of f' .

By these (1) and (2), we obtain equations of Diophantine type. By solving these equalities, we can derive various properties of X (see §6, §7 and §8).

In this paper we solve the equations in the following case:

Main Theorem (see Theorem 5.0 and Section 6). *Let X be a \mathbb{Q} -factorial \mathbb{Q} -Fano 3-fold with the following properties:*

- (1) $\rho(X) = 1$;
- (2) $I(X) = 2$;
- (3) $F(X) = \frac{1}{2}$;
- (4) $h^0(-K_X) \geq 4$;
- (5) *there exists an index 2 point P such that*

$$(X, P) \simeq (\{xy + f(z^2, u) = 0\}/\mathbb{Z}_2(1, 1, 1, 0), o)$$

with $\text{ord} f(Z, U) = 1$.

Then X is isomorphic to one in the following table:

Notation for the tables. Let $f : Y \rightarrow X$ be the weighted blow up at P with weight $\frac{1}{2}(1, 1, 1, 2)$. We will also use the notation as above explanation.

$$h := h^0(-K_X).$$

$N := \text{aw}(X)$ (see Definition 1.1 for the definition of $\text{aw}(X)$).

e is defined as follows:

If there is a flop while $Y \dashrightarrow Y'$, then $e := E^3 - E_1^3$, where E_1 is the strict transform of E on Y_1 (we will know that $e > 0$). If there is no flop while $Y \dashrightarrow Y'$, then $e := 0$.

$n := \sum \text{aw}(Y_i, P_{ij})$, where the summation is taken over the index 2 points on flipping curves.

z and u is defined as follows: If f' is birational, then let E' be the exceptional divisor of f' and set $E' \equiv z(-K_{Y'}) - u\tilde{E}$ or if f' is not birational, then let L be the pull back of an ample generator of $\text{Pic}X'$ and set $L \equiv z(-K_{Y'}) - u\tilde{E}$, where \tilde{E} is the strict transform of E on Y' .

In case f' is of type E_1 , then let $C := f'(E')$.

$$l_C := (-K_{X'} \cdot C).$$

In case f' is of type G , let Δ be the discriminant divisor of f'

In case f' is of type D , let F be a general fiber of f' .

Q_3 means the smooth 3-dimensional quadric.

B_i ($1 \leq i \leq 5$) means the \mathbb{Q} -factorial Gorenstein terminal Fano 3-fold X with $\rho(X) = 1$, $F(X) = 2$ and $(-K_X)^3 = 8i$.

V_{2i} ($1 \leq i \leq 11$ and $i \neq 10$) means the \mathbb{Q} -factorial Gorenstein terminal Fano 3-fold X with $\rho(X) = 1$, $F(X) = 1$ and $(-K_X)^3 = 2i$.

Type [i] means the \mathbb{Q} -Fano 3-fold of type [i] which was classified by T.Sano in [San2].

The mark \bigcirc means that there is an example.

$h = 4$							
exists ?	$(-K_X)^3$	N	e	n	z	l_C	f', X'
\bigcirc	$\frac{5}{2}$	1	15	0	1	/	$E_5, (-K_{X'})^3 = \frac{5}{2}, I(X') = 2$
?	$\frac{5}{2}$	1	15	0	1	/	crep. div., $(-K_{X'})^3 = 2, I(X') = 1$
\bigcirc	3	2	12	1	1	/	E_9, V_4
	$\frac{7}{2}$	3	10	0	1	1	E_1, V_6
	4	4	8	0	1	2	E_1, V_8
	4	4	9	3	1	/	E_2, V_{10}
	$\frac{9}{2}$	5	6	0	1	3	E_1, V_{10}
	$\frac{9}{2}$	5	12	3	1	/	E_6, V_{16}
	$\frac{9}{2}$	5	9	0	2	/	$D, \deg F = 6$
	5	6	4	0	1	4	E_1, V_{12}
	5	6	8	1	2	/	$D, \deg F = 8$

$$z = u.$$

$h = 5$								
exists ?	$(-K_X)^3$	N	e	n	z	$\deg \Delta$	$\deg F$	f'
\bigcirc	$\frac{9}{2}$	1	9	0	1	/	3	D
	5	2	8	1	1	/	4	D
	$\frac{11}{2}$	3	7	2	1	/	5	D
	$\frac{11}{2}$	3	8	0	2	8	/	$C, \mathbb{F}_{2,0}$
	6	4	7	1	2	6	/	$C, \mathbb{F}_{2,0}$
	6	4	6	3	1	/	6	D
	$\frac{13}{2}$	5	6	2	2	4	/	$C, \mathbb{F}_{2,0}$
	7	6	5	3	2	2	/	$C, \mathbb{F}_{2,0}$
	$\frac{15}{2}$	7	4	4	2	0	/	$C, \mathbb{F}_{2,0}$

$$h = 6$$

exists ?	$(-K_X)^3$	N	e	n	z	$\deg \Delta$	l_C	f', X'
○	$\frac{13}{2}$	1	7	0	1	7	/	C, \mathbb{P}^2
○	7	2	7	0	4	/	35	$E_1, [5]$
	7	2	6	1	1	6	/	C, \mathbb{P}^2
○	$\frac{15}{2}$	3	7	0	2	/	9	$E_1, [2]$
○	$\frac{15}{2}$	3	6	1	4	/	30	$E_1, [5]$
	$\frac{15}{2}$	3	5	2	1	5	/	C, \mathbb{P}^2
	8	4	4	3	1	4	/	C, \mathbb{P}^2
	$\frac{17}{2}$	5	3	4	1	3	/	C, \mathbb{P}^2
	9	6	2	5	1	2	/	C, \mathbb{P}^2
	$\frac{19}{2}$	7	1	6	1	1	/	C, \mathbb{P}^2

$u = z$ in case f' is of type C .

$$h = 7$$

exists ?	$(-K_X)^3$	N	e	n	z	l_C	f', X'
○	$\frac{17}{2}$	1	6	0	3	36	E_1, \mathbb{P}^3
○	9	2	6	0	2	18	$E_1, [3]$
○	9	2	5	1	3	32	E_1, \mathbb{P}^3
○	$\frac{19}{2}$	3	5	1	2	15	$E_1, [3]$
○	$\frac{19}{2}$	3	4	2	3	28	E_1, \mathbb{P}^3

$u = z + 1$.

$$h = 8$$

exists ?	$(-K_X)^3$	N	e	n	z	l_C	f, X'
○	$\frac{21}{2}$	1	6	0	1	6	E_1, B_3
○	$\frac{21}{2}$	1	5	0	2	27	E_1, Q_3
○	11	2	4	1	2	24	E_1, Q_3
	$\frac{23}{2}$	3	3	2	2	21	E_1, Q_3

$u = z + 1$.

$$h = 9$$

exists ?	$(-K_X)^3$	N	e	n	z	u	l_C	f', X'
○	$\frac{25}{2}$	1	5	0	1	2	10	E_1, B_4

$$h = 10$$

exists ?	$(-K_X)^3$	N	e	n	$\deg \Delta$	l_C	f', X'
○	$\frac{29}{2}$	1	4	0	/	14	E_1, B_5
	$\frac{29}{2}$	1	6	0	0	/	C, \mathbb{P}^2
○	15	2	3	1	/	12	E_1, B_5

$z = 1$ and $u = 2$.

□

Based on this result, we can derive the following properties for X as in the main theorem if X has only $\frac{1}{2}(1, 1, 1)$ singularities as its non-Gorenstein points:

Theorem A (See Corollary 7.2). $| -K_X |$ has a member with only canonical singularities.

So the general elephant conjecture by Miles Reid is affirmative for X .

Theorem B (See Corollary 8.1). Let $f : Y \rightarrow X$ be as in the main theorem, i.e., f is the blow up at a $\frac{1}{2}(1, 1, 1)$ -singularity and $g : Y \rightarrow Z$ the anti-canonical model (by the main theorem, g is found to be not an isomorphism). Then if $N > 1$ (resp. $N = 1$), Z can be deformed to a \mathbb{Q} -Fano 3-fold Z' with $\rho(Z') = 1$ and $F(Z') = \frac{1}{2}$ which has only $N - 1$ $\frac{1}{2}(1, 1, 1)$ -singularities as its singularities and $h^0(-K_{Z'}) = h$ (resp. a smooth Fano 3-fold Z' with $\rho(Z') = 1$, $F(Z') = 1$ and $h^0(-K_{Z'}) = h$.)

This is an analogue to the Reid's fantasy about Calabi-Yau 3-folds [RM3].

Theorem C (See Corollary 8.3). X can be embedded into a weighted projective space $\mathbb{P}(1^h, 2^N)$, where $h := h^0(-K_X)$ and N is the number of $\frac{1}{2}(1, 1, 1)$ -singularities on X .

We hope that this fact can be used for the classification of Mukai's type (see [Mu1], [Mu2] and [Mu3]).

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Notation and Conventions.

- (1) In this paper, we will work over \mathbb{C} , the complex number field;
- (2) we denote the linear equivalence by \sim and the numerical equivalence by \equiv . The equality $=$ in an adjunction formula means the \mathbb{Q} -linear equivalence;
- (3) we denote the Hirzebruch surface of degree n by \mathbb{F}_n and the surface which is obtained by the contraction of the negative section of \mathbb{F}_n by $\mathbb{F}_{n,0}$.

1. PRELIMINARIES

Theorem 1.0 (Vanishing theorem). Let $f : X \rightarrow Y$ be a projective morphism from a normal variety X with only Kawamata log terminal singularities. Let D be a \mathbb{Q} -Cartier integral Weil divisor such that $D - K_X$ is f -nef and f -big. Then $R^i f_* \mathcal{O}_X(D) = 0$ for all $i > 0$.

We will quote this theorem as *KKV vanishing theorem*.

Proof. See [Kod1], [KY1] and [V]. \square

Definition 1.1. Let (X, P) be a germ of 3-dimensional terminal singularity of index > 1 . By the classification of such a singularity [Mo2], we can easily see that a general deformation of (X, P) has only cyclic quotient singularities. We call the number of these cyclic quotient singularities the *arithmetic weight* of (X, P) and denote

it by $\text{aw}(X, P)$. Let X be a 3-fold with only terminal singularities. We define $\text{aw}(X) := \sum \text{aw}(X, P)$, where the summation takes place over points of index > 1 .

Theorem 1.2 (Special case of the singular Riemann-Roch Theorem). *Let X be a 3-fold with at worst index 2 terminal singularities and D an integral Weil divisor on X . Then the following formula holds:*

$$\chi(\mathcal{O}_X(D)) = \chi(\mathcal{O}_X) + \frac{1}{12}D(D - K_X)(2D - K_X) + \frac{1}{12}D.c_2(X) + \sum c_Q(D),$$

where the summation takes place over non Gorenstein points where D is not Cartier and $\sum c_Q(D) = -\frac{n}{8}$ for some non negative integer n . (See [RM2, Theorem 10.2] for the definition of $c_Q(D)$.)

Proof. See [RM2, Theorem 10.2]. \square

Theorem 1.3. *Let X be a projective 3-fold with at worst index 2 terminal singularities. Then $-K_X.c_2(X) = 24 - \frac{3N}{2}$, where $N := \text{aw}(X)$. Furthermore assume that X is a \mathbb{Q} -factorial \mathbb{Q} -Fano 3-fold with $\rho(X) = 1$. Then $-K_X.c_2(X) > 0$. In particular $N \leq 15$.*

Proof. See [KY2, Lemma 2.2 and Lemma 2.3] and [KY3, Proposition 1]. \square

Corollary 1.4. *Let X be a weak \mathbb{Q} -Fano 3-fold with $I(X) = 2$. Then $h^0(-K_X) = 3 + \frac{1}{2}(-K_X)^3 - \frac{N}{4}$, where $N := \text{aw}(X)$.*

Proof. This directly follows from Theorem 1.0, Theorem 1.2 and Theorem 1.3. \square

Proposition 1.5. *Let $f : X \rightarrow (Y, Q)$ be a flopping contraction from a 3-fold X with only terminal singularities to a germ (Y, Q) and $f^+ : X^+ \rightarrow Y$ the flop of f constructed as in [Kol1, Theorem 2.4]. Let D be a Cartier divisor on X and D^+ the strict transform of D on X^+ . Then D^+ is a Cartier divisor.*

Proof. By passing to the analytic category and taking algebraization [Ar, Theorem 3.8], we may assume that $C := \text{excep}f$ is irreducible. Furthermore since we can deform X to a 3-fold with only cyclic quotient terminal singularities [Mo3, (1b.8.2) Corollary] and such a deformation lifts to one of $f : X \rightarrow Y$ [KoMo, (11.4) Proposition], we may assume that X has only cyclic quotient terminal singularities. Let H' be a general hyperplane section through Q and $H := f^*H'$. Then it is well known that

(1.5.1) H' and H have only canonical singularities and H is dominated by the minimal resolution of H' .

We show that there are at most 2 singularities on C . Assume the contrary. Then there are 3 singularities on C and they coincide singularities of H on C by (1.5.1). Let $p : \tilde{Y} \rightarrow Y$ be the canonical cover, $\tilde{X} := X \times_Y \tilde{Y}$, \tilde{C} (resp. \tilde{H}' , \tilde{H}) the pull back of C (resp. H' , H) on \tilde{X} and $\tilde{f} : \tilde{X} \rightarrow \tilde{Y}$ the induced morphism. Then \tilde{X} is smooth and \tilde{f} is also a flopping contraction. We will prove that \tilde{C} is irreducible. If Q is not of the exceptional type ([RM2, Theorem (6.1) (2)]) and \tilde{C} is reducible, then there are components which intersect at 3 points, a contradiction to $R^1\tilde{f}_*\mathcal{O}_{\tilde{X}} = 0$. Hence \tilde{C} is irreducible in this case. If Q is of the exceptional type and \tilde{C} is reducible, then \tilde{C} has two component \tilde{C}_1 , \tilde{C}_2 and they intersect at one point transversely. But $\tilde{C}_i \rightarrow C$ is a double cover between \mathbb{P}^1 's and is branched at three points, a contradiction to Hurwitz's formula. Hence in any case \tilde{C} is irreducible. By [RM2,

(4.10)], \tilde{H} must be smooth. Hence \tilde{H}' has only ODP whence H' has a canonical singularity of type A. But then H has at most 2 singularities, a contradiction. So we have the assertion.

Furthermore H has exactly two singularities. For otherwise $\text{aw}(Y, Q) = 1$ since $\text{aw}(Y, Q) = \text{aw}(X)$. Hence Q is a cyclic quotient singularity but then there is no flopping contraction to Q , a contradiction.

We can prove as above that \tilde{C} is irreducible if Q is not of the exceptional type or \tilde{C} has at most 2 components if Q is of the exceptional type.

Assume that Q is not of the exceptional type. Let r be the index of Q . Let P be a non Gorenstein point on C and \tilde{P} the inverse image on \tilde{X} . Then P is also of index r and by [RM2, (4.10)], we have locally analytically

$$(\tilde{P} \in \tilde{C} \subset \tilde{X}) \simeq (o \in \{x = y = 0\} \subset \mathbb{C}^3),$$

where x, y, z are coordinates of \mathbb{C}^3 which are semi-invariants of \mathbb{Z}_r -action. Let \tilde{E} be a Cartier divisor which is localized to $z = 0$ and E the image of \tilde{E} on X . Then we have $E.C = \frac{1}{r}$. Since rE is a Cartier divisor and $\text{Pic}X \simeq \text{Pic}C$, we have $D \sim r(D.C)E$. Then we have $D^+ \sim r(D.C)E^+$, where E^+ is the strict transform of E on X^+ because linear equivalence is preserve by a flop. Since the analytic types of X and X^+ are the same by [Kol1, Theorem 2.4], $r(D.C)E^+$ is Cartier and so is D^+ .

Assume that Q is of the exceptional type. Then X has one index 2 point and one index 4 point. Using [Kol2, Proposition 2.2.6], we can determine Q as follows:

$$(Y, Q) \simeq \{x^2 - y^2 + (z^2 - u^{4k+4})(z - au^{4l+2})\}/\mathbb{Z}_4(1, 3, 2, 1), o),$$

where k and l are non negative integers and $a \in \mathbb{C}$. Let $\tilde{f}_1 : \tilde{X}_1 \rightarrow \tilde{Y}$ be the blow up along $\{x + y = z + u^{2k+2} = 0\}$ and $\tilde{f}_2 : \tilde{X}_2 \rightarrow \tilde{X}_1$ the blow up along the strict transform of $\{x - y = z - u^{2k+2} = 0\}$. Let \tilde{F}_1 (resp. \tilde{F}_2) be the strict transform of $\{x + y = z + u^{2k+2} = 0\}$ (resp. $\{x - y = z - u^{2k+2} = 0\}$) on \tilde{Y}_2 . Then $-\tilde{F}_1 - \tilde{F}_2$ is $\tilde{f}_1 \circ \tilde{f}_2$ -ample and $(\tilde{f}_1 \circ \tilde{f}_2)_*(\tilde{F}_1 + \tilde{F}_2)$ is \mathbb{Z}_4 -invariant. Hence \mathbb{Z}_4 acts on \tilde{X}_2 regularly and $\tilde{f}_1 \circ \tilde{f}_2$ is equivariant, i.e., we can identify $\tilde{f} : \tilde{X} \rightarrow \tilde{Y}$ and $\tilde{f}_1 \circ \tilde{f}_2 : \tilde{X}_2 \rightarrow \tilde{Y}$. Let $F' := p_*\{x + y = z + u^{2k+2} = 0\}_{\text{red}}$ and F its strict transform on X . We can see that F is a Cartier divisor and $F.C = -1$. Note that the involution $(x, y, z, u) \rightarrow (-x, y, z, u)$ induce an involution ι on Y . Then we have $F' + \iota_*F' \sim 0$ whence F^+ is a Cartier divisor. Since $D \sim -(D.C)F$ by $\text{Pic}X \simeq \text{Pic}C$, D^+ is also a Cartier divisor.

□

2. EXTREMAL CONTRACTIONS FROM 3-FOLDS WITH ONLY INDEX 2 TERMINAL SINGULARITIES

Definition 2.0 (Extremal contraction). Let X be an analytic 3-fold with only terminal singularities and $f : X \rightarrow (Y, Q)$ a projective morphism onto a germ of a normal variety with only connected fibers. Let except be the locus where f is not isomorphic. Assume that $-K_X$ is f -ample.

- (1) If $\dim Y = 3$ and $\dim \text{except} = 1$, then we say that f is an extremal contraction of flipping type (or in short a flipping contraction).

- (2) Only in this case, we assume that $-K_X$ is f -numerically trivial instead that $-K_X$ is f -ample. If $\dim Y = 3$ and $\dim \text{excep} f = 1$, then we say that f is a flopping contraction.
- (3) Assume that $\dim Y = 3$, $\text{excep} f$ is purely 2-dimensional and every component of the exceptional divisor E is contracted to a curve. Let $C := f(E)$. Assume furthermore that over a general point of every component of C , f coincides with the blow up along C and $-E$ is f -ample. Then we say that f is an extremal contraction of type E_1 .
- (4) Assume that $\dim Y = 3$, $\text{excep} f$ is an irreducible divisor E and $f(E)$ is a point. Then we say that f is an extremal contraction of type $E_{\geq 2}$.
- (5) If $\dim Y = 2$ and every fiber is 1-dimensional, then we say that f is an extremal contraction of type C .
- (6) If $\dim Y = 1$ and $f^{-1}(Q)_{\text{red}}$ is irreducible, then we say that f is an extremal contraction of type D .

Proposition 2.1 (Flipping contraction). *Let X be an analytic 3-fold with only index 2 terminal singularities and $f : X \rightarrow (Y, Q)$ a flipping contraction to a germ (Y, Q) . Let C be its exceptional curve. (Since (Y, Q) is a germ, C is connected.) Then*

- (1) $C \simeq \mathbb{P}^1$ and there is only one index 2 singularity on C and $-K_X.C = \frac{1}{2}$;
- (2) let P be the unique index 2 singularity on C . Then locally analytically $(P \in C \subset X) \simeq (o \in \{x_2 = x_3 = x_4 = 0\} \subset \{x_1x_2 + p(x_3^2, x_4) = 0\}/\mathbb{Z}_2(1, 1, 1, 0))$;
- (3) let $p(0, x_4) = ax_4^k$, where a is a unit in $\mathbb{C}\{x_1, x_2, x_3, x_4\}$ and $k \in \mathbb{N}$ (note that $k = aw(X, P)$). Then there is a deformation $\mathfrak{f} : \mathfrak{X} \rightarrow \mathfrak{Y}$ of f over a 1-dimensional disk $(\Delta, 0)$ such that for $t \neq 0$, \mathfrak{X}_t has only $k\frac{1}{2}(1, 1, 1)$ -singularities and $\mathfrak{f}_t : \mathfrak{X}_t \rightarrow \mathfrak{Y}_t$ is a bimeromorphic morphism which is localized to k flipping contractions.
- (4) assume that P is a $\frac{1}{2}(1, 1, 1)$ -singularity. Then we can construct the flip of f as follows:

Let $g : X_1 \rightarrow X$ be the blow up of P and E_1 the exceptional divisor. Let $h : X_2 \rightarrow X_1$ be the blow up along the strict transform C_1 of C on X_1 and E_2 the exceptional divisor. Then $E_2 \simeq \mathbb{P}^1 \times \mathbb{P}^1$ and we can blow it down to another direction. Let $i : X_2 \rightarrow X_1^+$ be the blow down and E_1^+ the strict transform of E_1 on X_1^+ . Then $E_1^+ \simeq \mathbb{F}_1$ and we can blow it down to the ruling direction. Let $j : X_1^+ \rightarrow X^+$ be the blow down. Then $X \dashrightarrow X^+$ is the flip.

- (5) If X is projective and f is an algebraic flipping contraction, then $(-K_{X^+})^3 = (-K_X)^3 - \frac{n}{2}$, where $n = \sum aw(X, P)$ and the summation is taken over the non Gorenstein points on flipping curves.

Proof. As for (1), (2) and (4), see [KoMo, (4.2) and (4.4.5)]. We will prove (3). Construct Y' as in [ibid. (4.3)]. Then $Y' = \{y_1y_3 + y_2p(y_2^2, y_4) = 0\}$ as in [ibid. (4.4.2)]. Then f is obtained by blowing-up of Y' along $\{y_2 = y_3 = 0\}$ and dividing by the \mathbb{Z}_2 action. Let $\mathfrak{Y}' = \{y_1y_3 + y_2(p(y_2^2, y_4) + ty_4) = 0\}$ be a deformation of Y' over a 1-dimensional disk $(\Delta, 0)$. Then by blowing-up of \mathfrak{Y}' along $\{y_2 = y_3 = 0\}$ and dividing by the induced \mathbb{Z}_2 action, we obtain the desired \mathfrak{f} . Next we prove (5). If we compactify \mathfrak{X} in (3), then (5) holds by (4) and the invariance of $(-K)^3$ in a flat family. Since $(-K_X)^3 - (-K_{X^+})^3$ can be expressed by an intersection number

of the pull back of $(-K_X)$ with exceptional divisors on a simultaneous resolution of X^+ and X (and hence it is determined locally around flipping curves), the general case follows. \square

Proposition 2.2 (Contraction of type E_1). *Let X be an analytic 3-fold with only index 2 terminal singularities and $f : X \rightarrow (Y, Q)$ an extremal contraction of type E_1 to a germ (Y, Q) . Let E be the exceptional divisor and $C := f(E)$. Let l be the fiber over Q . Then the following holds:*

- (1) *Assume that l contains no index 2 point. Then Q is a smooth point and f is the blow up along C ;*
- (2) *Assume that l contains an index 2 point. Then l contains only one index 2 point (we will denote it by P) and every component l' of l passes through P and satisfies $-K_X.l' = \frac{1}{2}$.*

Assume furthermore that X is projective and $\rho(X/Y) = 1$. Then the following formula holds:

$$(-K_E)^2 = 8(1 - g(\overline{C})) - 2m,$$

where \overline{C} is the normalization of C and m is a non-negative integer.

Proof. See [Mo1, Theorem 3.3] for (1). Assume that X is projective and $\rho(X/Y) = 1$. Let $\mu : \overline{E} \rightarrow E$ be the normalization and define a \mathbb{Q} -divisor Z by $K_{\overline{E}} = \mu^* K_E - Z$. Then Z is effective and its support is contained in fibers. Hence $Z.(-K_{\overline{E}}) \geq 0$ and $(-K_E)^2 \leq (-K_{\overline{E}})^2 \leq 8(1 - g(\overline{C}))$. Since $-K_X - E \sim f^*(-K_Y) - 2m$, $(-K_E)^2 = (-K_X - E)^2 E = 2(2E^3 - 2f^*(-K_Y)E^2) \in 2\mathbb{Z}$. Hence we have the formula as above. \square

Proposition 2.3 (Contraction of type $E_{\geq 2}$). *Let X be a 3-fold with only index 2 terminal singularities and $f : X \rightarrow (Y, Q)$ a divisorial contraction to a germ (Y, Q) which contracts a divisor E to Q . Then the following holds:*

- (1) *Assume that E contains no index 2 point. Then one of the following holds:*

$$(E_2) : (E, -E|_E) \simeq (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)) \text{ and } Q \text{ is a smooth point};$$

$$(E_3) : (E, -E|_E) \simeq (\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^3}(1)|_{\mathbb{P}^1 \times \mathbb{P}^1}) \text{ and } (Y, Q) \simeq (((xy + zw = 0) \subset \mathbb{C}^4), o);$$

$$(E_4) : (E, -E|_E) \simeq (\mathbb{F}_{2,0}, \mathcal{O}_{\mathbb{P}^3}(1)|_{\mathbb{F}_{2,0}}) \text{ and } (Y, Q) \simeq (((xy + z^2 + w^k = 0) \subset \mathbb{C}^4), o) (k \geq 3);$$

$$(E_5) : (E, -E|_E) \simeq (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2)) \text{ and } Q \text{ is a } \frac{1}{2}(1, 1, 1)\text{-singularity.}$$

Furthermore for all cases, f is the blow up of Q .

- (2) *Assume that E contains an index 2 point. Then one of the following holds:*

$(E_6) : (E, -E|_E) \simeq (\mathbb{F}_{2,0}, l)$, where l is a ruling of $\mathbb{F}_{2,0}$.

Q is a smooth point and f is a weighted blow up with weight $(2, 1, 1)$.

In particular we have $K_X = f^*K_Y + 3E$;

$(E_7) : K_X = f^*K_Y + E$ and Q is a Gorenstein singular point. $E^3 = \frac{1}{2}$;

$(E_8) : K_X = f^*K_Y + E$ and Q is a Gorenstein singular point. $E^3 = 1$;

$(E_9) : K_X = f^*K_Y + E$ and Q is a Gorenstein singular point. $E^3 = \frac{3}{2}$;

$(E_{10}) : K_X = f^*K_Y + E$ and Q is a Gorenstein singular point. $E^3 = 2$;

$(E_{11}) : (E, -E|_E) \simeq ((\{xy + w^2 = 0\} \subset \mathbb{P}(1, 1, 2, 1)), \mathcal{O}(2))$.

$(Y, Q) \simeq (((xy + z^k + w^2 = 0) \subset \mathbb{C}^4/\mathbb{Z}_2(1, 1, 0, 1)), o)$.

f is a weighted blow up with a weight $(\frac{1}{2}, \frac{1}{2}, 1, \frac{1}{2})$.

In particular we have $K_X = f^*K_Y + \frac{1}{2}E$;

$(E_{12}) : (E, -E|_E) \simeq (\mathbb{F}_{2,0}, 3l)$.

Q is a $\frac{1}{3}(2, 1, 1)$ -singularity and f is a weighted blow up with a weight $\frac{1}{3}(2, 1, 1)$.

In particular we have $K_X = f^*K_Y + \frac{1}{3}E$;

Proof. See [Mo1, Theorem 3.4 and Corollary 3.5], [Cu] for (1) and [Lu, Corollary 2.5 and Theorem 2.6] for (2) and Q is a non Gorenstein point. We will prove here that if Q is a Gorenstein point, f is of type $E_6 \sim E_{10}$. Let a be the discrepancy for E . Since Q is assumed to be Gorenstein, a is a positive integer.

First assume that $a \geq 2$. Let $L := -2E$. Then L is free by [AW] since $K_X + \frac{a}{2}L \equiv 0$ and $\frac{a}{2} \geq 1$. Let D be a general member of $|L|$ and $C := E|_D$. Since $-K_D \equiv -(a-2)E|_D$ is nef and big, C is a tree of \mathbb{P}^1 by KKV vanishing theorem. Let $\mu : \tilde{E} \rightarrow E$ be the normalization of E . If C is reducible, then μ^*C is not connected, a contradiction to the ampleness of μ^*C . Hence $C \simeq \mathbb{P}^1$. By this we know that E is normal since E satisfies S_2 condition. Since C is ample and isomorphic to \mathbb{P}^1 , $E \simeq \mathbb{P}^2, \mathbb{F}_n (n \geq 1)$ or $\mathbb{F}_{n,0} (n \geq 2)$ by a classical result (see for example [Ba]). But if former 2 cases occur, X is smooth, a contradiction to the assumption of (2). Hence $E \simeq \mathbb{F}_{n,0} (n \geq 2)$. We will prove that $n = 2$. Let v be the vertex of E . Then v is the unique singularity on E and hence it is of index 2. If E is Cartier at v , then for a exceptional divisor E over v with discrepancy

$\frac{1}{2}$, the discrepancy of F for K_Y is not an integer, a contradiction. Hence $K_X + E$ is a Cartier divisor and hence K_E is Cartier at v . So n must be 2. Furthermore by $K_E = (a+1)E|_E$, $a = 3$ since $a \geq 2$ and $E \simeq \mathbb{F}_{2,0}$. By taking the canonical cover near v of X , we know that v is a $\frac{1}{2}(1, 1, 1)$ -singularity. We will prove that Q is smooth and f is a weighted blow up with a weight $(2, 1, 1)$. We see that \overline{E} is contracted to a curve and let $\overline{X} \rightarrow \overline{X'}$ the contraction. Then next we can contract the strict transform of F to a smooth point, which is no other than Q . We can easily show that a weighted blow up with a weight $(2, 1, 1)$ is decomposed into contractions as above. So we are done.

Next we assume that $a = 1$. Let P be an index 2 point on X . If E is Cartier at P , then for a exceptional divisor F over P with discrepancy $\frac{1}{2}$, the discrepancy of F for K_Y is not an integer, a contradiction. Hence E is not Cartier at P whence $M := -K_X - E$ is an ample Cartier divisor. So E is a Gorenstein (possibly non normal) del Pezzo surface since $-K_E = M|_E$. Since $\chi(\mathcal{O}_E) = 1$ by [Sak, Theorem (5.1)] and [RM5, Corollary 4.10], $\text{Pic}E$ is torsion free. So $-K_X + E|_E \sim 0$ and hence $-K_X + E \sim 0$ by $\text{Pic}X \simeq \text{Pic}E$. So we note that $M \sim -2K_X$. Since $(-K_E)^2 = 4E^3 \geq 2$, $|-K_E|$ is free by [RM5, Corollary 4.10] and [Fu2, Corollary 1.5]. By the exact sequence

$$0 \rightarrow \mathcal{O}_X(-2E - K_X) \rightarrow \mathcal{O}_X(-E - K_X) \rightarrow \mathcal{O}_E(-K_E) \rightarrow 0$$

and the KKV vanishing theorem, $|M|$ is also free. Let G be a general member of $|M|$, $l := E|_G$ and $G' := f(G)$. Then Q is a minimally elliptic singularity of G' by the formula $K_G = f|_G^* K_{G'} - l$ and [La, Theorem 3.4]. On the other hand, the embedded dimension of Q is at most 4 since Q is a cDV singularity on Y . Hence we have $-(l^2)_G \leq 4$ by [La, Theorem 3.13] whence $(-K_E)^2 = -2(l^2)_G = 2, 4, 6, 8$. These correspond to type $E_7 \sim E_{10}$ respectively. \square

Proposition 2.4 (Contraction of type C). *Let X be an analytic 3-fold with only index 2 terminal singularities and $f : X \rightarrow (Y, Q)$ an extremal contraction of type C to a germ of surface. Let l be the fiber over Q . Then Q is a smooth point or an ordinary double point. Furthermore the following description holds:*

- (1) *if l contains no index 2 point, Q is a smooth point and f is a usual conic bundle;*
- (2) *if l contains an index 2 point and Q is a smooth point, l contains only one index 2 point and every component l' of l passes through it. Furthermore $-K_X.l' = \frac{1}{2}$;*
- (3) *if l contains an index 2 point and Q is an ordinary double point, f is analytically isomorphic to one of the following:*
- (3-1) *Let $\mathbb{P}^1 \times (\mathbb{C}^2, o) \rightarrow (\mathbb{C}^2, o)$ be the natural projection. Define the action of the group \mathbb{Z}_2 on $\mathbb{P}^1_{x_0, x_1} \times \mathbb{C}^2_{u, v}$:*

$$(x_0, x_1; u, v) \rightarrow (x_0, -x_1; -u, -v).$$

Let $X = \mathbb{P}^1 \times \mathbb{C}^2 / \mathbb{Z}_2$ and $(Y, Q) = (\mathbb{C}^2 / \mathbb{Z}_2, o)$.

In particular X has two $\frac{1}{2}(1, 1, 1)$ -singularities on l and $l_{\text{red}} \simeq \mathbb{P}^1$ and $-K_X.l_{\text{red}} = 1$.

- (3-2) *Let X' be a hypersurface in $\mathbb{P}^2_{x_0, x_1, x_2} \times \mathbb{C}^2_{u, v}$ defined by the equation $x_0^2 + x_1^2 + x_2^2 + \phi(u, v) = 0$, where $\phi(u, v)$ has no multiple factors and contains*

only monomials of even degree. Let $f' : X' \rightarrow \mathbb{C}^2$ be the natural projection. Define the action of the group \mathbb{Z}_2 on X' as follows:

$$(x_0, x_1, x_2; u, v) \rightarrow (-x_0, x_1, x_2; -u, -v).$$

Let $X := X'/\mathbb{Z}_2$ and $(Y, Q) = (\mathbb{C}^2/\mathbb{Z}_2, o)$.

In particular P is the unique index 2 point and $\text{aw}(X, P) = 2$. If $\text{mult}_{(0,0)}(\phi) = 2$, then (X, P) is a $cA/2$ point or if $\text{mult}_{(0,0)}(\phi) \geq 4$, then (X, P) is a $cAx/2$ point.

Proof. See [Mo1, Theorem 3.5] for (1) and [Pr, Theorems 3.1, 3.15 and Examples 2.1 and 2.3] for (2) and (3). \square

Proposition 2.5 (Contraction of type D). *Let X be an analytic 3-fold with only index 2 terminal singularities and $f : X \rightarrow (C, Q)$ be an extremal contraction of type D to a germ of a curve. Let F be the fiber over Q . Then Q is a smooth point and the following description holds:*

- (1) *if F contains no index 2 point, then all fibers are irreducible and reduced and (possibly non normal) Gorenstein del Pezzo surfaces. Furthermore if $(-K_F)^2 = 9$, we can write $-K_X \sim 3A$ for some relatively ample divisor A and $X = \mathbb{P}(f_*\mathcal{O}_X(A))$ which is a \mathbb{P}^2 -bundle;
if $(-K_F)^2 = 8$, we can write $-K_X \sim 2A$ for some relatively ample divisor A and X is embedded in \mathbb{P}^3 -bundle $\mathbb{P}(f_*\mathcal{O}_X(A))$ as a quadric bundle (the last means all fibers are quadrics in \mathbb{P}^3);
the case $(-K_F)^2 = 7$ does not occurs.*
- (2) *if F contains an index 2 point, then F is irreducible and reduced or $F = 2F_{\text{red}}$ and F_{red} is irreducible. F_{red} is a del Pezzo surface of Gorenstein index 2.*

Proof. See [Mo1, Theorem 3.5] for (1). (2) follows from the existence of a section [Co]. \square

3. TAKEUCHI'S THEORY

Definition 3.0. Let X be a \mathbb{Q} -Fano variety. We say that a birational morphism $f : Y \rightarrow X$ is a weak \mathbb{Q} -Fano blow up if the following hold:

- (1) Y is a weak \mathbb{Q} -Fano variety;
- (2) f is an extremal divisorial contraction such that f -exceptional divisor is a prime \mathbb{Q} -Cartier divisor.

In this section, we consider a \mathbb{Q} -factorial \mathbb{Q} -Fano 3-fold X with the following properties:

Assumption 3.1.

- (1) the Picard number $\rho(X)$ is 1;
- (2) there is a weak \mathbb{Q} -Fano blow up $f : Y \rightarrow X$. Let E be the f -exceptional divisor.

We fix f as in Assumption 3.1 and set $\alpha := (-K_Y)^2 E$.

Consider the extremal ray R of Y other than the ray associated to f . If R is a ray associated to a non-small contraction, denote by $f' : Y \rightarrow X'$ the contraction

associated to R . If R is a flopping ray, then after the flop $Y_0 := Y \xrightarrow{g_0} Y_1$, another extremal ray of Y_1 is K_{Y_1} -negative because K_{Y_1} is not nef and $\rho(Y_1) = 2$. Hence we can start the minimal model program from Y or Y_1 and because the canonical bundle can not become nef while the program, we obtain the following diagram:

$$\begin{array}{ccc} Y & \dashrightarrow & Y' \\ f \swarrow & & \searrow f' \\ X & & X' \end{array},$$

where $Y \dashrightarrow Y'$ is an isomorphism or a composition of possibly one flop and flips and f' is the first non small contraction. Let E be the exceptional divisor of f . We do the similar calculations as Kiyohiko Takeuchi did in [T1] in the below. The following lemma is basic for our computations:

Lemma 3.2. *Assume that $Y \dashrightarrow Y'$ is not an isomorphism. Let*

$$Y_0 := Y \xrightarrow{g_0} Y_1 \dots \xrightarrow{g_{k-1}} Y_k := Y'$$

be the decomposition of $Y \dashrightarrow Y'$ into flops and flips. Let l_i be an irreducible component of the flipping (or flopping) curve for g_i and E_i the strict transform of E on Y_i . Then

- (1) *there is at most one flop in the above decomposition and if there is, the flop is g_0 ;*
- (2) *$E_i.l_i > 0$;*
- (3) *if $Y \xrightarrow{g_0} Y_1$ is a flop, then $(-K_{Y_1})^3 = (-K_Y)^3$, $(-K_{Y_1})^2 E_1 = (-K_Y)^2 E$, $(-K_{Y_1}) E_1^2 = (-K_Y) E^2$ and $e := E^3 - E_1^3 \in \frac{\mathbb{N}}{s}$, where s is the minimum positive integer such that sE is a Cartier divisor;*
- (4) *if $Y_i \xrightarrow{g_i} Y_{i+1}$ is a flip, let $d_i := (-K_{Y_i})^3 - (-K_{Y_{i+1}})^3$. Then $d_i > 0$ and $(-K_{Y_{i+1}})^2 E_{i+1} = (-K_{Y_i})^2 E_i - a_i d_i$, $(-K_{Y_{i+1}}) E_{i+1}^2 = (-K_{Y_i}) E_i^2 - a_i^2 d_i$ and $E_{i+1}^3 = E_i^3 - a_i^3 d_i$, where $a_i := \frac{E_i.l_i}{(-K_{Y_i}).l_i}$ (note that this number a_i is well defined since flipping curves are numerically proportional);*
- (5) *if $Y_i \xrightarrow{g_i} Y_{i+1}$ and $Y_{i+1} \xrightarrow{g_{i+1}} Y_{i+2}$ are flips, then $a_{i+1} < a_i$;*
- (6) *$\text{Pic} Y'$ is torsion free.*

Proof.

- (1) This is clear from the above consideration;
- (2) we prove this by induction for i . For $i = 0$, assume that $E_0.l_0 \leq 0$. Then E_0 is non positive for two extremal rays of Y and hence E_0 is non positive for all effective curves on Y since $\rho(Y) = 2$. But this contradicts the effectivity of E . Assume that $E_i.l_i > 0$ is proved. Then $E_{i+1}.l_i^+ < 0$, where l_i^+ is the flipped curve corresponding to l_i . Hence we can prove $E_{i+1}.l_{i+1} > 0$ by the same way as proving $E_0.l_0 > 0$;
- (3) let

$$\begin{array}{ccc} & Z & \\ p \swarrow & & \searrow q \\ Y & & Y_1 \end{array},$$

the common resolution of Y and Y_1 . Then by the negativity lemma ([FA, Lemma 2.10]), we can easily see that $p^*K_Y = q^*K_{Y_1}$ (for example, see

[Kol1, Proof of Lemma 4.3] or below argument). By this, former 3 equalities follows. We prove that $e \in \frac{\mathbb{N}}{s}$. Since sE_1 is Cartier by Proposition 1.5, we have $e \in \frac{\mathbb{Z}}{s}$. Let

$$p^{-1}E = p^*E - R = q^*E_1 - R',$$

where R and R' are effective divisors which are exceptional for p and q . Rewrite this as

$$-p^*E = -q^*E_1 + R' - R.$$

Then since $-q^*E_1$ is p -nef by (3), we see that $R' - R > 0$ and $p_*(R' - R) \neq 0$ by $E.l_0 > 0$ and the negativity lemma. Hence we can write $p^*E = q^*E_1 - F$, where $F := R' - R$ is an effective divisor. Consider the identity $(p^*E)(q^*E_1)^2 = (q^*E_1 - F)(q^*E_1)^2$. Its right side is equal to E_1^3 . Its left side is equal to $(p^*E)(p^*E + F)^2 = E^3 + E.p_*(F^2)$. By $p_*F \neq 0$, we know that $-p_*(F^2)$ is a non zero effective 1-cycle. Hence $E.p_*(F^2) < 0$ and we are done;

- (4) the proof is very similar to one of (4). Let

$$\begin{array}{ccc} & Z & \\ p \swarrow & & \searrow q \\ Y_i & & Y_{i+1} \end{array} ,$$

the common resolution of Y_i and Y_{i+1} . By the definition of a_i ,

$$(a) \quad H_i := a_i(-K_{Y_i}) - E_i$$

is numerically trivial for the flipping curves. Let H_i^+ be the strict transform of H_i . By the negativity lemma, we can easily see that $p^*H_i = q^*H_i^+$ and $p^*(-K_{Y_i}) = q^*(-K_{Y_{i+1}}) - G$, where G is an effective divisor which is exceptional for p and q . $d_i > 0$ can be proved similarly to the proof of positivity of e . Consider the following identities:

$$(b) \quad (-K_{Y_i})^2 H_i = (p^*(-K_{Y_i}))^2 p^*H_i = (q^*(-K_{Y_{i+1}}) - G)^2 q^*H_i^+ = (-K_{Y_{i+1}})^2 H_i^+$$

and similarly

$$(c) \quad (-K_{Y_i})H_i^2 = (-K_{Y_{i+1}})H_i^{+2}$$

and

$$(d) \quad H_i^3 = H_i^{+3}.$$

By (a)~(d) and the definition of d_i , we obtain the assertion;

- (5) let l_i^+ be a flipped curve on Y_{i+1} . By $(a_i(-K_{Y_{i+1}}) - E_{i+1}).l_i^+ = 0$ and $(a_i(-K_{Y_{i+1}}) - E_{i+1}).m > 0$ for a general curve m on Y_{i+1} , we have $(a_i(-K_{Y_{i+1}}) - E_{i+1}).l_{i+1} > 0$. On the other hand we have $(a_{i+1}(-K_{Y_{i+1}}) - E_{i+1}).l_{i+1} = 0$. Hence we are done;

- (6) it is easy to see by Riemann-Roch theorem that $\text{Pic}Y$ is torsion free since Y is a weak \mathbb{Q} -Fano 3-fold. Since $Y \dashrightarrow Y'$ is a composition of a flop or flips and linear equivalence is preserved under a flop and a flip, $\text{Pic}Y'$ is also torsion free.

□

We will define e , all a_i 's and n_i 's to be 0 if $Y = Y'$. If $Y \dashrightarrow Y'$ is not an isomorphism, we will define e to be 0 if $Y \dashrightarrow Y_1$ is not a flop and a_i and n_i to be 0 if $Y_i \dashrightarrow Y_{i+1}$ is not a flip. From now on, we divide f' into cases. For this, we have

Claim. *If f' is a crepant contraction, then $Y = Y'$ and $\dim X' = 3$.*

Proof. The fact that $Y = Y'$ is clear by consideration above Lemma 3.2. Since $-K_Y$ is a supporting divisor of f' and $-K_Y$ is nef and big, $\dim X' = 3$. □

Hence we have the following cases:

Case 1. f' is an extremal contraction of type E_1 .

Case 2. f' is an extremal contraction of type $E_2 \sim E_{11}$.

Case 3. f' is an extremal contraction of type C .

Case 4. f' is an extremal contraction of type D .

Case 5. f' is a crepant divisorial contraction.

Claim 3.3. *\tilde{E} and $-K_{Y'}$ are numerically independent.*

Proof. For Y' , the numerical equivalence is equal to the \mathbb{Q} -linear equivalence by Lemma 3.2 (6). So the assertion follows since no multiple of \tilde{E} moves and $-K_{Y'}$ is big. □

In Case 1, 2 or 5, let E' be the exceptional divisor of f' , \tilde{E} the strict transform of E on Y' and \tilde{E}' the strict transform of E' on Y . By Claim 3.3 and $\rho(Y') = 2$, we can write

$$(3-0-1) \quad E' \equiv z(-K_{Y'}) - u\tilde{E}.$$

In Case 3 or 4, let L be the pull back of the ample generator of $\text{Pic}X'$ and \tilde{L} the strict transform of L on Y . By Claim 3.3 and $\rho(Y') = 2$, we can write

$$(3-0-2) \quad L \equiv z(-K_{Y'}) - u\tilde{E}.$$

Assumption 3.4. In the below we further assume that $P := f(E)$ is a point of index r and $-K_Y = f^*(-K_X) - \frac{1}{r}E$ and write $-K_X \equiv qS$, where S is the positive generator of $Z^1(X)/\equiv$ and q is a positive integer.

Then we have

Claim 3.5. *$z \in \mathbb{N}/q$ and u is a positive rational number such that $z + ru \in \mathbb{N}$.*

Proof. On X , $f(\tilde{E}') \equiv zqS$ in Case 1, 2 or 5 (resp. $f(\tilde{L}) \equiv zqS$ in Case 3 or 4).

So by Assumption 3.4, $z \in \frac{\mathbb{N}}{q}$.

If $u \leq 0$, sufficient multiple of E' must be move in Case 1, 2 or 5 (resp. L must be big in Case 2 or 3), a contradiction.

Let \tilde{E}' be the strict transform of E' on Y . By (3-0-1) and $-K_Y = f^*(-K_X) - \frac{1}{r}E$, we have $\tilde{E}' \equiv zf^*(-K_X) - (\frac{z}{r} + u)E$ in Case 1, 2 or 5 (resp. by (3-0-2) and $-K_Y = f^*(-K_X) - \frac{1}{r}E$, we have $\tilde{L} \equiv zf^*(-K_X) - (\frac{z}{r} + u)E$ in Case 3 or 4). Hence $\frac{z}{r} + u \in \frac{\mathbb{N}}{r}$. □

Case 1. Let $C := f'(E')$.

Claim 3.6. $z + 1 = uk$ for some $k \in \mathbb{N}$.

Proof. By (3-0-1) and $-K_{Y'} = f'^*(-K_{X'}) - E'$, we have $(z+1)E' \equiv zf'^*(-K_{X'}) - u\tilde{E}$. Since $f'(\tilde{E})$ is a Cartier divisor along C outside a finite set of points, $\frac{z+1}{u}$ is an integer. \square

We have the following:

Recall that $\alpha := (-K_Y)^2 E$.

$$(3-1-1) \quad (-K_{Y'} + E')^2(-K_{Y'}) = (z+1)^2(-K_{Y'})^3 - 2u(z+1)(-K_{Y'})^2\tilde{E} + u^2(-K_{Y'})\tilde{E}^2 = (-K_{X'})^3.$$

$$(3-1-2) \quad (-K_{Y'} + E')^2 E' = z(z+1)^2(-K_{Y'})^3 - u(z+1)(3z+1)(-K_{Y'})^2\tilde{E} + u^2(3z+2)(-K_{Y'})\tilde{E}^2 - u^3\tilde{E}^3 = 0.$$

$$(3-1-3) \quad (-K_{Y'} + E')E'(-K_{Y'}) = (z+1)z(-K_{Y'})^3 - u(2z+1)(-K_{Y'})^2\tilde{E} + u^2(-K_{Y'})\tilde{E}^2 = (-K_{X'}.C).$$

$$(3-1-4) \quad (-K_{Y'} - E')^2 E' = 4\{(-K_{X'})^3 - (-K_{Y'})^3 - 2(-K_{X'}.C)\} \\ z(z-1)^2(-K_{Y'})^3 - u(z-1)(3z-1)(-K_{Y'})^2\tilde{E} + u^2(3z-2)(-K_{Y'})\tilde{E}^2 - u^3\tilde{E}^3 = (-K_{E'})^2 \leq 8(1 - g(\overline{C})),$$

where \overline{C} is the normalization of C . (the last inequality of (3-1-4) can be proved similarly to the proof of Proposition 2.2.)

Hence by Lemma 3.2, we obtain the following:

$$(3-1-1') \quad \{k^2(-K_Y)^3 - (2k+r)\alpha - \sum d_i(a_i - k)^2\}u^2 = (-K_{X'})^3.$$

$$(3-1-2') \quad (uk-1)k^2(-K_Y)^3 - \{ur^2 + (3uk-1)r + k(3uk-2)\}\alpha + \sum d_i\{u(a_i-k)^3 + (a_i-k)^2\} + eu = 0.$$

$$(3-1-3') \quad \{k(uk-1)(-K_Y)^3 - (2uk-1+ur)\alpha - \sum d_i(a_i-k)(a_iu-ku+1)\}u = (-K_{X'}.C).$$

The positivity of the left hand side gives some information. By (3-1-1') and (3-1-2'), we have the following:

$$(3-1-5') \quad e + \sum d_i a_i (a_i - k)^2 = (r+k)^2 \alpha - \frac{uk-1}{u^3} (-K_{X'})^3.$$

The following claim is useful for solving the equations:

Claim 3.7. *If $Y_i \dashrightarrow Y_{i+1}$ is a flip, then $k < a_i$.*

Proof. Note that $f'^*(-K_{X'}) \equiv u\{k(-K_{Y'}) - \tilde{E}\}$. Hence $k(-K_{Y'}) - E_i$ is \mathbb{Q} -effective for any i . If $Y_i \dashrightarrow Y_{i+1}$ is a flip and $k \geq a_i$ for some i , then $(k(-K_{Y'}) - E_i).l_i \geq 0$ and hence $(k(-K_{Y_{i+1}}) - E_{i+1}).l_i^+ \leq 0$. By the \mathbb{Q} -effectivity of $k(-K_{Y_{i+1}}) - E_{i+1}$ and $\rho(Y_{i+1}) = 2$, $k(-K_{Y_{i+1}}) - E_{i+1}$ is positive for another extremal ray of Y_{i+1} . So $k(-K_{Y'}) - \tilde{E}$ is positive for a fiber of f' . But this is absurd. \square

Case 2. We note that $Y \neq Y'$ since otherwise $E \neq E'$ and the nonempty intersection curve $E \cap E'$ is contracted by two extremal contractions f and f' , a contradiction.

Let $\frac{d}{r'}$ be the discrepancy of E' for $K_{X'}$, where r' is the index of $P' := f'(E')$. By the similar way to the proof of Claim 3.6, we have the following claim:

Claim 3.8. *$zd + r' = uk$ for some $k \in \mathbb{N}$.*

Proof. By (3-0-1) and $-K_{Y'} = f'^*(-K_{X'}) - \frac{d}{r'}E'$, we have $(zd + r')E' \equiv r'zf'^*(-K_{X'}) - ur'\tilde{E}$. Since $r'f'(\tilde{E})$ is Cartier divisor at P' , $\frac{zd+r'}{u}$ is an integer. \square

We have the following:

$$(3-2-1) \quad z^3(-K_{Y'})^3 - 3z^2u(-K_{Y'})^2\tilde{E} + 3zu^2(-K_{Y'})\tilde{E}^2 - u^3\tilde{E}^3 = (E')^3.$$

$$(3-2-2) \quad z^2(-K_{Y'})^3 - 2zu(-K_{Y'})^2\tilde{E} + u^2(-K_{Y'})\tilde{E}^2 = (-K_{Y'})(E')^2.$$

$$(3-2-3) \quad z(-K_{Y'})^3 - u(-K_{Y'})^2\tilde{E} = (-K_{Y'})^2E'.$$

Hence by Lemma 3.2, we obtain the following:

$$(3-2-1') \quad z^3(-K_Y)^3 - u\alpha(u^2r^2 + 3zur + 3z^2) + \sum d_i(ua_i - z)^3 + u^3e = (E')^3.$$

$$(3-2-2') \quad z^2(-K_Y)^3 - u\alpha(2z + ur) - \sum d_i(ua_i - z)^2 = (-K_{Y'})(E')^2.$$

$$(3-2-3') \quad z(-K_Y)^3 - u\alpha + \sum d_i(ua_i - z) = (-K_{Y'})^2(E').$$

By (3-2-1') and (3-2-2'), we have the following:

$$(3-2-4') \quad \sum d_ia_i(a_iu - z)^2 + u^2e = \alpha(z + ur)^2 + \frac{k}{r'}(E')^3.$$

By (3-2-2') and (3-2-3'), we have the following:

$$(3-2-5') \quad \alpha(z + ur) + \sum d_ia_i(a_iu - z) = \frac{dk}{r'^2}(E')^3.$$

Similarly to Claim 3.7, we have the following:

Claim 3.9. *If $Y_i \dashrightarrow Y_{i+1}$ is a flip, then $k < da_i$.*

Proof. Note that $f'^*(-K_{X'}) \equiv \frac{u}{r'}\{k(-K_{Y'}) - d\tilde{E}\}$. Hence $k(-K_{Y_i}) - dE_i$ is \mathbb{Q} -effective for any i . The rest is similar to the proof of Claim 3.7. \square

Case 3. By [Pr, Lemma 1.10], X' has only cyclic quotient singularities. By the general theory of the conic bundle, $-4K_{X'} \equiv f'_*(-K_{Y'}^2) + \Delta$, where Δ is the discriminant divisor of f' . Hence $-K_{X'} \cdot A > 0$ for any ample divisor A on X' since $-K_{Y'}$ is big. Hence X' is a log del Pezzo surface with $\rho(X') = 1$.

We have the following:

$$(3-3-1) \quad L^3 = z^3(-K_{Y'})^3 - 3z^2u(-K_{Y'})^2\tilde{E} + 3zu^2(-K_{Y'})\tilde{E}^2 - u^3\tilde{E}^3 = 0.$$

$$(3-3-2) \quad z^2(-K_{Y'})^3 - 2zu(-K_{Y'})^2\tilde{E} + u^2(-K_{Y'})\tilde{E}^2 = (-K_{Y'})L^2.$$

$$(3-3-3) \quad z(-K_{Y'})^3 - u(-K_{Y'})^2\tilde{E} = (-K_{Y'})^2L.$$

We set $u = mz$ and $l = f'_*L$. By Lemma 3.2, we obtain the following:

$$(3-3-1') \quad (-K_Y)^3 - m\alpha(m^2r^2 + 3mr + 3) + \sum d_i(ma_i - 1)^3 + m^3e = 0.$$

$$(3-3-2') \quad z^2\{(-K_Y)^3 - m\alpha(2 + mr) - \sum d_ia_i(ma_i - 1)^2\} = 2l^2.$$

$$(3-3-3') \quad z\{(-K_Y)^3 - m\alpha + \sum d_i(ma_i - 1)\} = (-K_Y)^2L.$$

If l is free, then $(-K_{Y'})^2L = 8(1 - g(l)) - \Delta \cdot l + 4l^2$.

By (3-3-1') and (3-3-2'), we have the following:

$$(3-3-4') \quad z^2\{\sum d_ima_i(ma_i - 1)^2 + m^3e\} = z^2m\alpha(mr + 1)^2 - 2l^2.$$

Case 4. By the edge sequence of the Leray spectral sequence $0 \rightarrow H^1(X', \mathcal{O}_{X'}) \rightarrow H^1(Y', \mathcal{O}_{Y'})$ (exact) and $H^1(Y', \mathcal{O}_{Y'}) = 0$, we have $H^1(X', \mathcal{O}_{X'}) = 0$, i.e., $X' \simeq \mathbb{P}^1$. Hence $L = f'^*\mathcal{O}_{\mathbb{P}^1}(1)$

We calculate the following:

$$(3-4-1) \quad (-K_{Y'})L^2 = z^2(-K_{Y'})^3 - 2zu(-K_{Y'})^2\tilde{E} + u^2(-K_{Y'})\tilde{E}^2 = 0.$$

$$(3-4-2) \quad \tilde{E}L^2 = z^2(-K_{Y'})^2\tilde{E} - 2zu(-K_{Y'})\tilde{E}^2 + u^2\tilde{E}^3 = 0.$$

$$(3-4-3) \quad (-K_{Y'})^2 L = z(-K_{Y'})^3 - u(-K_{Y'})^2 \tilde{E} = \deg F,$$

where F is a general fiber of f' and $\deg F := (-K_F)^2$.

We set $u = mz$. We obtain the following:

$$(3-4-1') \quad (-K_X)^3 = \frac{\alpha}{r} + m\alpha(2 + mr) + \sum d_i(ma_i - 1)^2.$$

$$(3-4-2') \quad (mr + 1)^2 \alpha = \sum d_i a_i (ma_i - 1)^2 + m^2 e.$$

$$(3-4-3') \quad z\{m\alpha(1 + mr) + \sum d_i ma_i (ma_i - 1)\} = \deg F.$$

The following claim is useful for solving the equations:

Claim 3.10. *In Case 3 or 4, if $Y_i \dashrightarrow Y_{i+1}$ is a flip, then $ma_i > 1$.*

Proof. Note that $L \equiv z(-K_{Y'} - m\tilde{E})$. The proof is similar to the one of Claim 3.7. \square

Case 5. Since $-K_Y.l = 0$ and $E'.l = -2$ for a general fiber l of E' , we have $u(E.l) = 2$. By $(-K_Y)^2 E' = 0$, we have $z(-K_Y)^3 = u\alpha$.

By an additional geometric assumption that $|-K_Y - E| \neq \phi$, the relation of u and z is restricted as follows:

Claim 3.11. *If $|-K_Y - E| \neq \phi$, then $z \leq u$. Furthermore in Case 3, $m = 1$ or 2 and in Case 4, $m = 1$ or $m = 2$ and $F \simeq \mathbb{P}^1 \times \mathbb{P}^1$ or $m = \frac{3}{2}$ or 3 and $F \simeq \mathbb{P}^2$.*

Proof. By (3-0-1), we have

$$(a) \quad E' \equiv (z - u)(-K_{Y'}) + u(-K_{Y'} - \tilde{E})$$

in Case 1, Case 2 or case 5 (resp. by (3-0-2),

$$(b) \quad L \equiv (z - u)(-K_{Y'}) + u(-K_{Y'} - \tilde{E})$$

in Case 3 or Case 4). By the assumption, $|-K_{Y'} - \tilde{E}| \neq \phi$. Hence if $z > u$, sufficient multiple of E' must move by (a) (resp. if $z > u$, $\kappa(L)$ must be 3 by (b)), a contradiction. So $z \leq u$.

In Case 3, for a general fiber C , we have $\tilde{E}.C = \frac{2z}{u} \in \mathbb{N}$. So $\frac{2z}{u} = 1$ or 2 since $z \leq u$. In Case 4, let C be a (-1) -curve in F if $F \not\simeq \mathbb{P}^1 \times \mathbb{P}^1, \mathbb{P}^2$ or a ruling if $F \simeq \mathbb{P}^1 \times \mathbb{P}^1$ or a line if $F \simeq \mathbb{P}^2$. By calculating $\tilde{E}.C$, we obtain the assertion similarly to Case 3. \square

4. EXISTENCE OF A WEAK Q-FANO BLOW
UP FOR A Q-FANO 3-FOLD WITH $I(X) = 2$

In this section, we give a sufficient condition for the existence of a weak Q-Fano blow up for a Q-Fano 3-fold with $I(X) = 2$.

Theorem 4.0. *Let X be a weak Q-Fano 3-fold with only log terminal singularities. Assume the following:*

- (1) $I(X) \leq 2$;
- (2) *there are only a finite number of non Gorenstein points on X ;*
- (3) $(-K_X)^3 \geq 1$ and $h^0(-K_X) \geq 1$.

Then $|-2K_X|$ is free.

Proof. By replacing X by its anti-canonical model, we can assume that X is a Q-Fano 3-fold with only log terminal singularities. By [Am, Theorem 1.2], S has only log terminal singularities. By the exact sequence $0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_X(-2K_X) \rightarrow \mathcal{O}_S(-2K_X|_S) \rightarrow 0$ and $h^1(\mathcal{O}_X) = 0$, we have $|-2K_X|_S = |-2K_X||_S$ and $\text{Bs}|-2K_X| = \text{Bs}|-2K_X|_S$. Note that $-K_X|_S = K_S$. Hence it suffices to prove that $|K_S + K_S|$ is free. Assume that $|2K_S|$ is not free. Let y be a base point of $|K_S + K_S|$. Assume that y is worse than canonical. By [KT, Theorem 9], y is a cyclic quotient singularity of index 2. So Kawachi's invariant δ' defined in [KT] is $\frac{1}{2}$ at y . On the other hand by the assumption that $(-K_X)^3 \geq 1$, $K_S^2 \geq 2$ holds. So $K_S^2 > \delta_y$ holds (δ_y is defined in [KaMa]). But by (1), we have $K_S.C = -K_X.C \geq \frac{1}{2}$ for any curve C whence by [ibid.], y cannot be a base point of $|2K_S|$, a contradiction. So we may assume that S does not contain a non Gorenstein point of X by (2) and has only canonical singularities. Let $\mu : \tilde{S} \rightarrow S$ be the minimal resolution. Since $h^0(K_{\tilde{S}}) = h^0(K_S) = h^0(-K_X) \geq 1$, $|2K_{\tilde{S}}|$ is free by [Fr] and hence $|2K_S|$ is free, a contradiction again.

Hence $|K_S + K_S|$ is free and also $|-2K_X|$ is free. \square

Proposition 4.1. *Let X be a weak Q-Fano 3-fold with $I(X) = 2$ such that $|-2K_X|$ is free. Let P be an index 2 point such that there is no curve l through P such that $-K_X.l = 0$. Let $f : Y \rightarrow X$ an extremal divisorial contraction from a 3-fold with only terminal singularities such that*

- (1) *f -exceptional divisor is a prime Q-Cartier divisor. We call it E ;*
- (2) $P := f(E)$ and $-K_Y = f^*(-K_X) - \frac{1}{2}E$;
- (3) $(-K_Y)^3 > 0$.

Then Y is a weak Q-Fano 3-fold.

Proof. By the assumption that there is no curve l through P such that $-K_X.l = 0$, $\text{Bs}|-2K_X - P|$ is a finite set of points near P . So by $H^0(-2K_Y) \simeq H^0(\mathcal{O}(-2K_X) \otimes m_P)$, we know $-K_Y$ is nef. So by (3), it is also big and we are done. \square

5. SOLUTION OF THE EQUATIONS OF DIOPHANTINE
TYPE FOR A Q-FANO 3-FOLD WITH $I(X) = 2$

Theorem 5.0. *Let X be a Q-factorial Q-Fano 3-fold with the following properties:*

- (1) $\rho(X) = 1$;
- (2) $I(X) = 2$;
- (3) $E(X) = 1$.

- (4) $h^0(-K_X) \geq 4$;
 (5) there exists an index 2 point P such that

$$(X, P) \simeq (\{xy + f(z^2, u) = 0\}/\mathbb{Z}_2(1, 1, 1, 0), o)$$

with $\text{ord}f(Z, U) = 1$.

Let $f : Y \rightarrow X$ be the weighted blow up at P with weight $\frac{1}{2}(1, 1, 1, 2)$. Then Y is a weak \mathbb{Q} -Fano 3-fold with $I(Y) = 2$ (and hence we can run the program as in Section 3). We use the notation as in there. Then $z \leq u$ and there is at most one flip while $Y \dashrightarrow Y'$ and $a_i = 2$ for i such that $Y_i \dashrightarrow Y_{i+1}$ is a flip. Furthermore we figure out the solutions of equations in Section 3 as in the following tables:

See the tables in the main theorem for the explanation about the notation.

Table 1. f' is of type E_1 and $u = z + 1$

h	$(-K_X)^3$	N	e	n	z	l_C	X'	
$\bigcirc 6$	7	2	7	0	4	35	$[5]$	
$\bigcirc 6$	$\frac{15}{2}$	3	7	0	2	9	$[2]$	
$\bigcirc 6$	$\frac{15}{2}$	3	6	1	4	30	$[5]$	
$\bigcirc 7$	$\frac{17}{2}$	1	6	0	3	36	\mathbb{P}^3	
$\bigcirc 7$	9	2	6	0	2	18	$[3]$	
$\bigcirc 7$	9	2	5	1	3	32	\mathbb{P}^3	
$\bigcirc 7$	$\frac{19}{2}$	3	5	1	2	15	$[3]$	
$\bigcirc 7$	$\frac{19}{2}$	3	4	2	3	28	\mathbb{P}^3	
$\bigcirc 8$	$\frac{21}{2}$	1	6	0	1	6	B_3	
$\bigcirc 8$	$\frac{21}{2}$	1	5	0	2	27	10	Q_3
$\bigcirc 8$	11	2	4	1	2	24	Q_3	
8	$\frac{23}{2}$	3	3	2	2	21	Q_3	
$\bigcirc 9$	$\frac{25}{2}$	1	5	0	1	10	B_4	
$\bigcirc 10$	$\frac{29}{2}$	1	4	0	1	14	B_5	
$\bigcirc 10$	15	2	3	1	1	12	B_5	

Table 2. f' is of type E_1 and $z = u = 1$

$(-K_X)^3$	N	e	l_C	X'
$\frac{7}{2}$	3	10	1	V_6
4	4	8	2	V_8
$\frac{9}{2}$	5	6	3	V_{10}
5	6	4	4	V_{12}
$\frac{11}{2}$	7	2	5	V_{14}
6	8	0	6	V_{16}

$h = 4$ and $n = 0$.

Table 3. f' is of type $E_2 \sim E_{12}$

h	$(-K_X)^3$	N	e	n	type of f' and X'
$\bigcirc 4$	$\frac{5}{2}$	1	15	0	E_5 or E_{11} , $(-K_{X'})^3 = \frac{5}{2}$, $I(X') = 2$
$\bigcirc 4$	3	2	12	0	E_9, V_4
4	4	4	9	3	E_2, V_{10}
4	$\frac{9}{2}$	5	12	3	E_6, V_{16}

$$z = u = 1.$$

Table 4. f' is of type C

h	$(-K_X)^3$	N	e	n	$\deg \Delta$
5	$\frac{11}{2}$	3	8	0	8
5	6	4	7	1	6
5	$\frac{13}{2}$	5	6	2	4
5	7	6	5	3	2
5	$\frac{15}{2}$	7	4	4	0
$\bigcirc 6$	$\frac{13}{2}$	1	7	0	7
6	7	2	6	1	6
6	$\frac{15}{2}$	3	5	2	5
6	8	4	4	3	4
6	$\frac{17}{2}$	5	3	4	3
6	9	6	2	5	2
6	$\frac{19}{2}$	7	1	6	1
6	10	8	0	7	0
10	$\frac{29}{2}$	1	6	0	0

If $h = 5$, then $z = u = 2$ and $X' \simeq \mathbb{F}_{2,0}$.

If $h = 6$, then $z = u = 1$ and $X' \simeq \mathbb{P}^2$.

If $h = 10$, then $z = 1, u = 2$ and $X' \simeq \mathbb{P}^2$.

Table 5. f' is of type D

h	$(-K_X)^3$	N	e	n	$\deg F$
4	$\frac{9}{2}$	5	9	0	6
4	5	6	8	1	8
$\bigcirc 5$	$\frac{9}{2}$	1	9	0	3
5	5	2	8	1	4
5	$\frac{11}{2}$	3	7	2	5
5	6	4	6	3	6

$z = u = 2$ in case $h = 4$.

$z = u = 1$ in case $h = 5$.

If f' is a crepant divisorial contraction, then

$$h = 4, (-K_X)^3 = \frac{5}{2}, N = 1, z = 1 \text{ and } u = 2.$$

Remark. We discuss the geometric realization in Section 6.

Proof. By (4) and Corollary 1.4, we have $(-K_X)^3 > 2$. Furthermore $(-K_Y)^3 = (-K_X)^3 - \frac{1}{2} > 0$. Hence by Proposition 4.1, Y is a weak \mathbb{Q} -Fano 3-fold. We can easily check that $I(Y) = 2$ by calculating the weighted blow up

We run the program as in Section 3. By the assumption that $h^0(-K_X) \geq 4$ and the exact sequence

$$0 \rightarrow \mathcal{O}_Y(-K_Y - E) \rightarrow \mathcal{O}_Y(-K_Y) \rightarrow \mathcal{O}_E(1) \rightarrow 0,$$

we have $|-K_Y - E| \neq \emptyset$. Hence by Claim 3.11, we have $z \leq u$.

First assume that Case 5 occurs. Since $u \in \frac{\mathbb{N}}{2}$ and $E.l \in \mathbb{N}$, we have $u = \frac{1}{2}, 1, 2$ by $u(E.l) = 2$. Furthermore since $z(-K_Y)^3 = u$, $(-K_Y)^3 > \frac{3}{2}$ and $z \leq u$, we have $z = 1$, $u = 2$ and $(-K_Y)^3 = 2$. Hence we are done in this case.

Claim 5.1. E_i is a Cartier divisor for any i . In particular a_i is an even integer.

Proof. Assume that g_0 is a flop. By Proposition 1.5, E_1 is a Cartier divisor since E is a Cartier divisor. The latter half follows from Proposition 2.1 (1). If g_i is a flip, there is no non Gorenstein point on the flipped curves. Hence E_i is Cartier by induction for i . \square

Note that by Lemma 3.2 (5) and Claim 5.1, once we prove that $a_i = 2$ if $a_i > 0$, we see that there is at most one flip while $Y \dashrightarrow Y'$.

Case 1. In this case we first show that $F(X') \geq 1$.

In fact by (3-0-1), we have $-K_{X'} \equiv \frac{u}{z}f'(\tilde{E})$. In this, $f'(\tilde{E})$ is Cartier and $u \geq z$. Hence the assertion holds. Furthermore by [13] and [San2], $F(X') = 1, \frac{3}{2}, 2, \frac{5}{2}, 3$ or 4.

We note that by Proposition 2.2, we have $(-K_{E'})^2 = 8(1 - g(\overline{C})) - 2m$ with some non negative integer m .

By $z + 1 = uk$ and $z \leq u$, we have $z + 1 = u$ or $z = u = 1$.

First assume that $z + 1 = u$. Define $a \in \mathbb{Z}$ by the formula $f(\tilde{E}) = aH$, where H is a primitive Cartier divisor of X' . Then $F(X') = a\frac{z+1}{z}$. Hence $z = 1, 2, 3, 4$ and if $z = 1$, then $F(X') = 2$ or 4, if $z = 2$, then $F(X') = \frac{3}{2}$ or 3, if $z = 3$, then $F(X') = 4$, or if $z = 4$, then $F(X') = \frac{5}{2}$. But we will prove that the case that $z = 1$ and $F(X') = 4$ does not occur. For otherwise, let H' be the strict transform of f'^*H on Y . Then we have $-K_Y \equiv 2H' + E$ and hence $-K_X \equiv 2f(H')$, a contradiction to $F(X) = \frac{1}{2}$.

Assume $a_i \geq 4$ for some i . Note that $a_i u > z$ by $u \geq z$. By (3-1-5'), $e \leq (k+2)^2 - 2(4-k)^2 < 0$, a contradiction.

Set $n := \sum n_i$. We obtain the following:

$$(5-1-1) \quad (-K_X)^3 = \frac{9+n}{2} + \frac{1}{u^2}(-K_{X'})^3$$

obtained by (3-1-1').

$$(5-1-2) \quad e + n = 9 - \frac{u-1}{u^3}(-K_{X'})^3$$

obtained by (3-1-5').

$$(5-1-3) \quad (-K_{X'} \cdot C) = \frac{u-1}{u}(-K_{X'})^3 - (3+n)u$$

obtained by (3-1-1') and (3-1-3').

$$(5-1-4) \quad -2(-K_{X'})^3 + 4(-K_{X'}.C) + 2(-K_X)^3 - n + 3 = 4g(\overline{C}) + m$$

obtained by (3-1-4).

Use (5-1-4) for the bound of n .

By (3-0-1), we have $\tilde{E}.l = 1$ for a general fiber l of E' . If E' contain an index 2 point, then there is a component l' of a fiber such that $-K_{Y'}.l' = \frac{1}{2}$ by Proposition 2.2. So we have $\tilde{E}.l' = \frac{1}{2}$. But this contradicts the fact that \tilde{E} is a Cartier divisor. Hence E' contains no index 2 point. This fact and information from X' determine N . Hence we can easily figure out the solutions.

Next assume $z = u = 1$. By Claim 3.7 and Claim 5.1, $a_i \geq 4$ if $a_i > 0$. Assume that $a_i \geq 6$ for some i . By (3-1-5'), $e \leq (k+2)^2 - 3(6-k)^2 < 0$, a contradiction. Hence we must have $a_i = 4$ for all i such that $Y_i \dashrightarrow Y_{i+1}$ is a flip. By setting $n := \sum n_i$, we rewrite (1-1) \sim (1-4) as follows:

$$(5-1-1') \quad e + 8n = 16 - (-K_{X'})^3$$

obtained by (3-1-5').

$$(5-1-2') \quad (-K_X)^3 = 6 - \frac{1}{4}e - \frac{3}{2}n.$$

obtained by (3-1-2').

$$(5-1-3') \quad (-K_{X'}.C) = 6 - \frac{1}{2}e - 6n.$$

obtained by (5-1-2') and (3-1-3').

$$(5-1-4') \quad 2(-K_X)^3 - 5 = 4g(\overline{C}) + m.$$

By (5-1-3') and $(-K_{X'}.C) > 0$, we must have $n = 0$, i.e., there is no flip while $Y \dashrightarrow Y'$.

By (1-1) and (1-2), we deduce that $(-K_{X'})^3 = 16 - e > 0$. By (1-1) and (1-3), we have $(-K_{X'}.C) = \frac{1}{2}(-K_{X'})^3 - 2 > 0$. By these, $(-K_{X'})^3 = 6, 8, 10, 12, 14, 16$.

Claim 5.2. $h^0(-K_X) = 4$.

Proof. By $\tilde{E} \equiv -K_{Y'} - E'$, we have $E \equiv -K_Y - \tilde{E}'$, where \tilde{E}' is the strict transform of E' . Since $E - (-K_Y - \tilde{E}')$ is a Cartier divisor, we must have $E \sim -K_Y - \tilde{E}'$ by Lemma 3.2 (6). Hence $h^0(-K_Y - E) = 1$. But by the exact sequence

$$0 \rightarrow \mathcal{O}_Y(-K_Y - E) \rightarrow \mathcal{O}_Y(-K_Y) \rightarrow \mathcal{O}_E(-K_Y|_E) \rightarrow 0,$$

we have

$$h^0(-K_Y - E) + h^0(-K_Y) \leq h^0(-K_Y - E) + h^0(-K_Y|_E) + 4$$

So $h^0(-K_X) = 4$. \square

Hence we have $N = \frac{16-e}{2}$.

We will prove that X' is Gorenstein. Assume that X' is non Gorenstein. If $F(X') = 1$, then by [San2], $N - 1 \geq 8$, a contradiction. Since $(-K_{X'})^3 = 16 - e$, $F(X') > 1$ does not hold by [San1]. Hence X' is Gorenstein.

Next we prove that $F(X') = 1$. By $(-K_{X'})^3 = 16 - e$, we only have to disprove that $F(X') = 2$. Assume that $F(X') = 2$. Let H be the ample generator of $\text{Pic} X'$ and $H' := f'^*H$. This is a Cartier divisor on Y' and so is the strict transform H'' on Y since $n = 0$. Since $H'' \equiv \frac{1}{2}(-K_Y + \tilde{E}') \equiv (-K_Y) - \frac{1}{2}E$, we have $f^*f_*H'' = H'' + E$. By this, we know f_*H'' is a Cartier divisor on X ([KMM, Lemma 3-2-5 (2)]). On the other hand, $f_*H'' \equiv -K_X$ and so $F(X)$ must be an integer, a contradiction to the assumption of Theorem 5.0. Hence we have $F(X') = 1$.

So we obtain the solutions as in the table.

Case 2. By Proposition 2.3, we obtain the following data:

In the below equations, the right sides are the values of the left sides in case f' is of type E_i ($i = 2 \sim 12$).

$$(E')^3 = \begin{matrix} E_2 & E_3, E_4 & E_5 & E_6 & E_7 & E_8 & E_9 & E_{10} & E_{11} & E_{12} \\ 1 & 2 & 4 & \frac{1}{2} & \frac{1}{2} & 1 & \frac{3}{2} & 2 & 4 & \frac{9}{2} \end{matrix}.$$

$$(-K_{Y'})(E')^2 = \begin{matrix} E_2 & E_3, E_4 & E_5 & E_6 & E_7 & E_8 & E_9 & E_{10} & E_{11} & E_{12} \\ -2 & -2 & -2 & -\frac{3}{2} & -\frac{1}{2} & -1 & -\frac{3}{2} & -2 & -2 & -\frac{3}{2} \end{matrix}.$$

$$(-K_{Y'})^2 E' = \begin{matrix} E_2 & E_3, E_4 & E_5 & E_6 & E_7 & E_8 & E_9 & E_{10} & E_{11} & E_{12} \\ 4 & 2 & 1 & \frac{9}{2} & \frac{1}{2} & \frac{E_8}{2} & \frac{3}{2} & 2 & 1 & \frac{1}{2} \end{matrix}.$$

Assume that f' is of type E_2 . By (3-2-5'), we have $z + 2u \leq 2k$. On the other hand, we have $1 + 2z = uk \geq zk$. Hence $z = u = 1$ and $k = 3$. By (3-2-5') again, $\sum d_i a_i (a_i - 1) = 3$. Since $a_i \geq 2$ if $a_i > 0$, we have $a_i = 2$ if $a_i > 0$. By setting $n := \sum n_i$, we have $n = 3$. We can easily see that $e = 9$, $(-K_X)^3 = 4$ and $(-K_{X'})^3 = 10$. By the assumption (4), we have $N = 4$. This also prove that X' is Gorenstein and hence X' is V_{10} .

We will prove that f' cannot be of type E_3 or E_4 . Assume that f' is of type E_3 or E_4 . Similarly to the above case, we have $k = 2$ and $\sum d_1 a_i (a_i - 1) = 1$ using (3-2-5'). But by Claim 3.9, we have $a_i \geq 4$ if $a_i > 0$, a contradiction.

If f' is of type $E_5 \sim E_{11}$, then we can figure out the solution similarly.

By these we can obtain the solutions.

Case 3. By Proposition 2.4, X' has at worst ordinary double points as singularities. Hence $X' \simeq \mathbb{P}^2$ or $\mathbb{F}_{2,0}$ and $L = f'^* \mathcal{O}_{\mathbb{P}^2}(1)$ if $X' \simeq \mathbb{P}^2$ or $L = f'^*(\mathcal{O}_{\mathbb{P}^3}(1)|_{\mathbb{F}_{2,0}})$ if $X' \simeq \mathbb{F}_{2,0}$.

Assume $a_i \geq 4$ for some i . Note that $a_i u > z$ by $u \geq z$. By (3-3-2'), $m^2 e < (2m + 1)^2 - 2(4m - 1)^2 < 0$, a contradiction. Hence $a_i = 2$ for all i such that $Y_i \dashrightarrow Y_{i+1}$ is a flip.

By setting $n := \sum n_i$, we have the following

$$(5-3-1) \quad (-K_X)^3 = \frac{1}{2} + m(4m^2 + 6m + 3) - \frac{n}{2}(2m-1)^3 - m^3e.$$

$$(5-3-2) \quad mz^2\{(2m+1)^2 - n(2m-1)^2 - m^2e\} = 2l^2 = \frac{\mathbb{P}^2}{2} \frac{\mathbb{F}_{2,0}}{4}.$$

$$(5-3-3) \quad mz\{2(2m+1)(m+1) - 2n(2m-1)(m-1) - m^2e\} = 12 - \frac{\mathbb{P}^2}{\Delta.l} \frac{\mathbb{F}_{2,0}}{16 - \Delta.l}.$$

By Claim 3.11, we have $m = 1$ or 2 .

If $m = 2$, we can easily figure out the solution.

Assume that $m = 1$. Then we have 3 sequences of solutions:

- (1) $X' \simeq \mathbb{P}^2$, $z = 1$, $n + e = 7$, $\Delta.l = e$ and $h^0(-K_X) = \frac{25+n-N}{4}$;
- (2) $X' \simeq \mathbb{F}_{2,0}$, $z = 1$, $n + e = 5$, $\Delta.l = 4 + e$ and $h^0(-K_X) = \frac{29+n-N}{4}$;
- (3) $X' \simeq \mathbb{F}_{2,0}$, $z = 2$, $n + e = 8$, $\Delta.l = 2e - 8$ and $h^0(-K_X) = \frac{23+n-N}{4}$.

If $X' \simeq \mathbb{P}^2$ and Y' has an index 2 point (resp. If $X' \simeq \mathbb{F}_{2,0}$ and $\text{aw}(Y') > 2$), then there is a fiber containing a component l such that $-K_{Y'}.l = \frac{1}{2}$ by Proposition 2.4. But these cases does not occur. For otherwise we have $\tilde{E}.l = \frac{z}{2u} < 1$, a contradiction to that \tilde{E} is a Cartier divisor. Hence for (1) and (2) (resp. (3)), we have $N - n = 1$ (resp. $N - n = 3$) since $\text{aw}(Y') = \text{aw}(Y) - n = N - n - 1$. But if (2) and $N - n = 1$ hold, Y' must be Gorenstein, a contradiction to Proposition 2.4. Hence we figure out the solutions as in the table.

Case 4. Similarly to Case 3, we can prove that $a_i = 2$ for all i such that $Y_i \dashrightarrow Y_{i+1}$ is a flip using (3-4-2').

By setting $n := \sum n_i$, we rewrite (3-4-1')~(3-4-3') as follows:

$$(5-4-1) \quad (-K_X)^3 = \frac{1}{2} + 2m(m+1) + \frac{1}{2}n(2m-1)^2.$$

$$(5-4-2) \quad (2m+1)^2 = n(2m-1)^2 + m^2e.$$

$$(5-4-3) \quad z\{m(2m+1) + nm(2m-1)\} = \deg F.$$

By Claim 3.11, we have $m = 1, \frac{3}{2}, 2$ or 3 .

We can easily see that there is no solution for $m = \frac{3}{2}, 2$ or 3 .

If $m = 1$, then we have $n + e = 9$, $(-K_X)^3 = \frac{n+9}{2}$ and $z(3+n) = \deg F$. Since $h^0(-K_X) = 3 + \frac{9+n-N}{4} \geq 4$, we have $N - n = 1$ or 5 . If $N - n = 1$, then Y' is Gorenstein. Hence by the primitivity of L , $z = 1$. If $N - n = 5$ and $u = z = 1$ or 3 , $L \not\sim z(-K_X - \tilde{E})$ since the right side is not Cartier. By Riemann-Roch theorem,

$\chi(\mathcal{O}(L)) - \chi(\mathcal{O}(z(-K_{Y'} - \tilde{E}))) = \frac{1}{2}$, a contradiction. Hence if $N - n = 5$, then $z = 2$ and so $n = 0$ or 1 by $z(3 + n) = \deg F$.

We will prove $n \leq 3$. If $n = 4$, then $\deg F = 7$, a contradiction to Proposition 2.5. If $n = 5$, then $Y' \rightarrow X'$ is a quadric bundle over a \mathbb{P}^1 by Proposition 2.5. But then $(-K_{Y'})^3$ must be a multiple of 8, a contradiction. If $n = 6$, then $Y' \rightarrow X'$ is a \mathbb{P}^2 -bundle over a \mathbb{P}^1 by Proposition 2.5. But then $(-K_{Y'})^3$ must be 54, a contradiction.

Hence we obtain the solutions as in the table. \square

Now we complete the proof of Theorem 5.0.

In the next section, we give examples for the cases with mark \bigcirc in the table and prove the non-existence of the cases $N = 7, 8$ in Table 2 and $h = 6$ and $N = 8$ in Table 4. This will complete the proof of the main theorem.

6. COMPLETION OF THE PROOF OF THE MAIN THEOREM AND EXAMPLES

We state a theorem proved by T. Minagawa which we need.

Theorem 6.0 (T. Minagawa). *Let X be a \mathbb{Q} -Fano 3-fold (resp. weak \mathbb{Q} -Fano 3-fold) with $I(X) = 2$. Assume that there exists a smooth member of $|-2K_X|$. Then there exists a flat family $\mathfrak{f} : \mathfrak{X} \rightarrow (\Delta, 0)$ over a 1-dimensional disk $(\Delta, 0)$ such that $X \simeq \mathfrak{f}^{-1}(0)$ and $\mathfrak{f}^{-1}(t)$ is a \mathbb{Q} -Fano 3-fold (resp. a weak \mathbb{Q} -Fano 3-fold) with only ODP's, QODP's or $\frac{1}{2}(1, 1, 1)$ -singularities as its singularities for $t \in \Delta \setminus \{0\}$, where ODP (resp. QODP) means a singularity analytically isomorphic to $\{xy + z^2 + u^2 = 0 \subset \mathbb{C}^4\}$ (resp. $\{xy + z^2 + u^2 = 0\} \subset \mathbb{C}^4/\mathbb{Z}_2(1, 1, 1, 0)$.)*

Proof. See [Mi2, Theorem 2.4]. \square

From now on we assume that X is a Fano 3-fold as in the main theorem.

Table 1. First we construct examples for the case that f' is of type E_1 and $u = z + 1$. We can treat all the cases at a time except $h = 8$ and $N = 3$. We don't know whether the case that $h = 8$ and $N = 3$ occurs or not.

Let S be a smooth Cartier divisor in X' such that $S \equiv \frac{z}{z+1}(-K_{X'})$. We can take such a S by [San2, Remark 4.1]. S is a del Pezzo surface. We represent S as blowing up at r points of \mathbb{P}^2 in general position, where $r := e + n$. Let E_1, \dots, E_r be its exceptional curves and l the total transform of a line in \mathbb{P}^2 . Let $D := l + E_1 + \dots + E_n$ and $C := -K_{X'}|_S - D$. Then we will show that $|C|$ is free. By computing intersection numbers with (-1) -curves, we can check that C is nef in any case in the table except the case that $h = 8$ and $N = 3$. Let $M := C - K_S$. Check that $M^2 > 4$. Hence if C is not free, there is an effective divisor l such that $M.l = 1$ and $l^2 = 0$ whence $-K_S.l = 1$ by Reider's theorem [RI]. But $l.(K_S + l) = -1$ is a contradiction. So $|C|$ is free.

Hence we denote a general smooth curve in $|C|$ also by C . Let $f' : Y' \rightarrow X'$ be the blow up along C and E' the exceptional divisor. Let \tilde{E} be the strict transform of S and $B := 2(-K_{Y'}) - \tilde{E}$. Let E_i' be the inverse image of E_i for $i = 1 \dots n$. We will check that B is nef and big. Let A be a Cartier divisor numerically equivalent to $\frac{z+2}{z+1}(-K_{X'})$. Since $B \sim f'^*A - E'$, we have only to show $|A - C|$ is free. Since $A - S$ is free and $C \subset S$, $\text{Bs}|A - C| \subset S$. By the exact sequence

and the KKV vanishing theorem, we see that $\text{Bs}|A - C| = \text{Bs}|A|_S - C|$. We can check that $|A|_S - C|$ is free by the same way as the check of the freeness of $|C|$. We also know that $A|_S - C$ is numerically trivial only for E_i 's ($i = 1 \dots n$).

Hence B is free and E_i 's ($i = 1 \dots n$) are numerically trivial for B . In particular two extremal rays of Y' are generated by the class of a fiber of f' and the class of E_i '. Let R_1 be the extremal ray generated by the class of E_i '. Then we will show that $\text{Supp } R_1 = \cup E_i'$. Since $\tilde{E}.E_i' = -2 < 0$, $\text{Supp } R_1 \subset \tilde{E}$. By $-K_{Y'}.E_i' = -1 < 0$, it is enough to show that $\text{Bs}|-K_{Y'}||_{\tilde{E}} = \cup E_i'$. For this, we have only to see that $\text{Bs}|-K_{X'} - C||_S = \cup E_i$. By the exact sequence

$$0 \rightarrow \mathcal{O}_{X'}(-K_{X'} - S) \rightarrow \mathcal{O}_{X'}(-K_{X'}) \rightarrow \mathcal{O}_S(-K_{X'}|_S) \rightarrow 0$$

and the KKV vanishing theorem, we see that $\text{Bs}|-K_{X'} - C||_S = \text{Bs}|-K_{X'}|_S - C| = \text{Bs}|D|$. By the definition of D , it is clear that $\text{Bs}|D| = \cup E_i$. So we are done.

By this, R_1 is a flipped ray. Observe that $\mathcal{N}_{E_i'/Y'} \simeq \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$. Let $Y' \dashrightarrow Y_1$ be the inverse of the flip and E_i' the strict transform of E_i on Y_1 for $i = n+1 \dots r$. Then we can easily show that $\cup_{i=n+1}^r E_i$ coincides with the support of the flopped ray of Y_1 . Let $Y_1 \dashrightarrow Y$ be the inverse of the flop and E the strict transform of \tilde{E} on Y . Then $(E, -K_Y|_E) \simeq (\mathbb{P}^2, \mathcal{O}_Y(1))$ and we can contract it. Set X the target of the contraction. Then X is what we want.

Table 2.

N=8. We deny the case that $N = 8$. Assume that $N = 8$. By Theorem 6.0, we may assume that any index 2 point is a $\frac{1}{2}(1, 1, 1)$ -singularity or a QODP. In this case $Y = Y'$ holds since $e = 0$. By $F(X') = 1$ and the \mathbb{Q} -factoriality of X' , there exists a line l intersecting C . Let l' be the strict transform of l on Y . By $-K_Y.l' = -K_{X'}.l - E'.l'$ and the fact that $-K_Y$ is nef, we have $-K_Y.l' = 0$ and $E'.l' = 1$ or $-K_Y.l' = \frac{1}{2}$ and $E'.l' = \frac{1}{2}$. But the latter case does not occur since $e = 0$. In the former case $E \cap l' = \phi$ by $E.l' = 0$. Hence $K_X.f(l') = \frac{1}{2}$, which in turn show that for a \mathbb{Q} -Fano blow up whose center is an index 2 point on $f(l')$, the resulting weak \mathbb{Q} -Fano 3-fold is not a \mathbb{Q} -Fano 3-fold. But by the tables in Section 5, we again fall into table 2 for the new choice of a \mathbb{Q} -Fano blow up, a contradiction (the new e must be 0).

Table 3. If $h = 4$ and $N = 1$, then $(X, -K_X) \simeq (((5) \subset \mathbb{P}(1^4, 2)), \mathcal{O}(1))$ is an example.

If $h = 4$ and $N = 2$, then $(X, -K_X) \simeq (((3, 4) \subset \mathbb{P}(1^4, 2^2)), \mathcal{O}(1))$ is an example.

Table 4.

$h = 6$ and $N = 1$. About the case that $h = 6$ and $N = 1$, we know that an example exists by Corollary 8.1 below and the existence of the case that $h = 6$ and $N = 2$ (see Table 1).

$h = 6$ and $N = 8$. We will show that if $h = 6$ and $N = 8$, then $F(X) = 1$. So we will exclude this case.

In this case, f' is a \mathbb{P}^1 -bundle associated to some vector bundle \mathcal{E} of rank 2 on \mathbb{P}^2 . Let T be its tautological divisor. By the adjunction formula $-K_{Y'} \sim 2T - (c_1(\mathcal{E}) - 3)L$, we have $6 = (-K_{Y'})^3 = 8T^3 - 6c_1(\mathcal{E})^2 + 54$ and hence $c_1(\mathcal{E})$ is an even. Hence $H' := 3T - (\frac{3}{2}c_1(\mathcal{E}) - 4)L$ is an integral Cartier divisor. Note that $H' = -K_{Y'} + \frac{1}{2}\tilde{E}$. Hence for a flipped curve l' on some Y and the strict

transform H_i of H' on Y_i , we have $H_i \cdot l_i^+ = -2$. Hence the strict transform H of H' on Y is a Cartier divisor numerically equivalent to $-K_Y + \frac{1}{2}E$. Note that H is f -numerically trivial. So by [KMM, Lemma 3-2-3 (2)], $f(H)$ is a Cartier divisor and clearly numerically equivalent to $-K_X$.

Now we complete the proof of the main theorem.

We close this section after proving some corollaries to the main theorem.

Corollary 6.1. *Let X be a \mathbb{Q} -factorial \mathbb{Q} -Fano 3-fold with the following properties:*

- (1) $\rho(X) = 1$;
- (2) $I(X) = 2$;
- (3) $F(X) = \frac{1}{2}$;
- (4) $h^0(-K_X) \geq 4$.

Then $(-K_X)^3$ and $\text{aw}(X)$ are effectively bounded as in the main theorem.

Proof. By the main theorem and Theorem 6.0, we obtain the assertion since $(-K)^3$ and aw are invariant under a deformation. \square

Lemma 6.2. *Let X be a \mathbb{Q} -factorial \mathbb{Q} -Fano 3-fold with $\rho(X) = 1$, $I(X) = 2$ and $F(X) = \frac{1}{2}$ and $f : Y \rightarrow X$ a weak Fano blow up with $I(Y) = 2$ such that f -exceptional divisor E is contracted to a point by f . If $|-2K_Y|$ is free and $h^0(-K_Y - E) > 0$, then $H^0(\mathcal{O}_Y(-2K_Y)) \rightarrow H^0(\mathcal{O}_E(-2K_Y|_E))$ is surjective.*

Proof. We are inspired by [RM1, p.29, Step 4]. It suffices to prove that $h^1(\mathcal{O}_Y(-2K_Y - E)) = 0$. Take a general member $T \in |-2K_Y|$. Then by the exact sequence

$$0 \rightarrow \mathcal{O}_Y(-E) \rightarrow \mathcal{O}_Y(-2K_Y - E) \rightarrow \mathcal{O}_T(-2K_Y - E|_T) \rightarrow 0$$

and $h^i(\mathcal{O}_Y(-E)) = 0$ for $i = 1, 2$ (these vanishing easily follow from

$$0 \rightarrow \mathcal{O}_Y(-E) \rightarrow \mathcal{O}_Y \rightarrow \mathcal{O}_E \rightarrow 0$$

since by Proposition 2.3, $h^1(\mathcal{O}_E) = 0$), we obtain $h^1(\mathcal{O}_Y(-2K_Y - E)) = h^1(\mathcal{O}_T(-2K_Y - E|_T))$. By Serre duality, we have $h^1(\mathcal{O}_T(-2K_Y - E|_T)) = h^1(\mathcal{O}_T(2K_T - E|_T)) = h^1(\mathcal{O}_T(K_Y + E|_T))$. We prove that $h^1(\mathcal{O}_T(K_Y + E|_T)) = 0$ in the below. Take a member $F \in |-K_Y - E| \neq \emptyset$. Then since $\rho(X) = 1$ and $-K_X$ is a positive generator of $Z^1(X)/\equiv$, we can write $F = F' + rE$, where F' is a prime divisor and r is a nonnegative integer. Since $|-2K_Y|$ is free and T is general, we may assume that $F'|_T$ and $E|_T$ is irreducible. Note that $(F' + rE)|_T \cdot E|_T = (-K_Y - E) \cdot E \cdot (-2K_Y) > 0$ and $(E|_T)^2 < 0$. Hence if $r > 0$, for every integer $b \in [1, r]$, we have $(F'|_T + (r - b)E|_T) \cdot bE|_T > 0$, which means $F|_T$ is numerically 1-connected. So by [RC, Lemma 3], we have $H^0(\mathcal{O}_{F|_T}) \simeq \mathbb{C}$. Hence by the exact sequence

$$0 \rightarrow \mathcal{O}_T(-F|_T) \rightarrow \mathcal{O}_T \rightarrow \mathcal{O}_{F|_T} \rightarrow 0,$$

we have $h^1(\mathcal{O}_T(-F|_T)) = 0$ which is exactly what we want. \square

Corollary 6.3. *Let X be a \mathbb{Q} -factorial \mathbb{Q} -Fano 3-fold with the following properties:*

- (1) $\rho(X) = 1$;
- (2) $I(X) = 2$;
- (3) $F(X) = \frac{1}{2}$;
- (4) $h^0(-K_X) \geq 4$.

Then for any index 2 point P , there exists a smooth rational curve l through P such that $-K_X.l = \frac{1}{2}$.

Proof. First we treat the case that any index 2 point is of type as in Theorem 5.0 (5). By Table 1 ~ Table 5 and nonexistence of the case that $h = 4$ and $N = 8$ and the case that $h = 6$ and $N = 8$, e is positive or f' is a crepant divisorial contraction for any choice of an index 2 point P . Let $g : Y \rightarrow Z$ be the anti-canonical model. Let l' be a flopping curve if g is a flopping contraction or a general fiber of E' if g is a crepant divisorial contraction. Then by Lemma 6.2, $g(E) \simeq E$ whence $E.l' = 1$. Hence $l := f(l')$ is what we want.

Next we treat the general case. Let $f : \mathfrak{X} \rightarrow \Delta$ be a flat family as in Theorem 6.0. By [KoMo, Corollary 12.3.4], $\rho(\mathfrak{X}_t) = 1$ and furthermore by Table 1 ~ 5, [San1] and [San2], $\mathfrak{X}_t(t \neq 0)$ satisfies the assumptions of Theorem 5.0. Let P be an index 2 point on X and P_t an index 2 point on \mathfrak{X}_t which specializes to P . By the first part of this proof, there is a curve l_t on $\mathfrak{X}_t(t \neq 0)$ such that $l_t \simeq \mathbb{P}^1$, $P_t \in l_t$ and $-K_{\mathfrak{X}_t}.l_t = \frac{1}{2}$. Since there are only countably many components of relative Hilbert scheme $\text{Hilb}(\mathfrak{X}/\Delta)$, we may assume that they form a flat family over Δ . Furthermore by the properness of a component of relative Hilbert scheme, this family extends over 0. Let l be its fiber over 0. Then l is what we want. \square

7. EXISTENCE OF AN ANTI-CANONICAL DIVISOR WITH ONLY CANONICAL SINGULARITIES FOR A Q-FANO 3-FOLD WITH $I(X) = 2$

Proposition 7.0. *Let X be as in Theorem 5.0 and assume furthermore that X has only $\frac{1}{2}(1, 1, 1)$ -singularities as its non Gorenstein points. Then $|-K_Y|$ has no base curve containing a $\frac{1}{2}(1, 1, 1)$ -singularity. The similar assertion holds also for X .*

Proof. Consider the exact sequence

$$0 \rightarrow \mathcal{O}_Y(-K_Y - E) \rightarrow \mathcal{O}_Y(-K_Y) \rightarrow \mathcal{O}_E(-K_Y|_E) \rightarrow 0.$$

Assume that we prove

$$(7.1.1) \quad h^0(\mathcal{O}_Y(-K_Y - E)) = h - 3.$$

Then the map $H^0(\mathcal{O}_Y(-K_Y)) \rightarrow H^0(\mathcal{O}_E(-K_Y|_E))$ is surjective. Since $|-K_Y|_E|$ is free, we know that $|-K_Y|$ has no base curve intersecting E . By this, we know that $|-K_X|$ has no base curve through $f(E)$. Since $f(E)$ is any $\frac{1}{2}(1, 1, 1)$ -singularity, it means $|-K_X|$ has no base curve containing a $\frac{1}{2}(1, 1, 1)$ -singularity. By this, $|-K_Y|$ has also no base curve containing a $\frac{1}{2}(1, 1, 1)$ -singularity.

So it suffices to show that (7.1.1) holds. We note that (7.1.1) is equivalent to

$$(7.1.2) \quad h^0(\mathcal{O}_{Y'}(-K_{Y'} - \tilde{E})) = h - 3.$$

We will prove this using the data of the tables in Theorem 5.0.

Table 1. We have $-K_{Y'} - \tilde{E} \sim f'^*D$, where D is a primitive ample Weil divisor (we can easily see that the linear equivalent class of D is unique). Hence $h^0(-K_{Y'} - \tilde{E}) = h^0(D)$. $h^0(D) = h - 3$ is easy to see.

Table 2 or Table 3. We have $-K_{Y'} - \tilde{E} \sim E'$ whence $h^0(-K_{Y'} - \tilde{E}) = 1 = h - 3$.

Table 4. Since $-K_{Y'} - \tilde{E} - K_{Y'}$ is nef and big, we can compute $h^0(-K_{Y'} - \tilde{E})$ by Riemann-Roth theorem and we are done.

But if $h = 6$, then $L \sim -K_{Y'} - \tilde{E}$ and hence we can see that $h^0(-K_{Y'} - \tilde{E}) = h^0(L) = 3 = h - 3$ more easily.

Table 5. Since $-K_{Y'} - \tilde{E} - K_{Y'}$ is nef and big, we can compute $h^0(-K_{Y'} - \tilde{E})$ by Riemann-Roth theorem and we are done. But if $h = 5$, then $L \sim -K_{Y'} - \tilde{E}$ and hence we can see that $h^0(-K_{Y'} - \tilde{E}) = h^0(L) = 2 = h - 3$ more easily.

□

Proposition 7.1. *Let X be a weak \mathbb{Q} -Fano 3-fold with log terminal singularities and satisfies the following condition:*

- (1) $|-K_X| \neq \emptyset$;
- (2) *there are a finite number of non Gorenstein points on X ;*
- (3) *there is a member of $|-K_X|$ which is normal near non Gorenstein points.*

Then $|-K_X|$ has a member which is normal and has only canonical singularities outside non Gorenstein points of X .

Proof. The proof is almost the same as one of [Am, Main Theorem]. So we only give an outline of the proof. Let $U := \{x | x \text{ is a Gorenstein point of } X\}$. Let S be a general member of $|-K_X|$. Let $\gamma := \max\{t | K_X + tS|_U \text{ is log canonical}\}$. As Ambro did, it suffices to prove that there is no element of $\text{CLC}(K_X + \gamma S|_U)$ contained in $\text{Bs}|-K_X|$. Assume the contrary and let Z be a minimal element of $\text{CLC}(K_X + \gamma S|_U)$ contained in $\text{Bs}|-K_X|$. By the assumption (3), Z is a complete variety. Hence by using Theorem 1.0, we know that it suffices to prove $H^0(\mathcal{O}_Z(-K_X|_Z)) \neq 0$. It is done by Adjunction Theorem and a nonvanishing argument. □

Corollary 7.2. *Let X be as in Theorem 5.0 and assume furthermore that X has only $\frac{1}{2}(1, 1, 1)$ -singularities as its non Gorenstein points. Then $|-K_X|$ has a member with only canonical singularities.*

Proof. Fix a $\frac{1}{2}(1, 1, 1)$ -singularity P and the blow up $f : Y \rightarrow X$ at P . By Proposition 7.0 and Proposition 7.1, we can find a member $S \in |-K_Y|$ such that S is normal and has only canonical singularities outside $\frac{1}{2}(1, 1, 1)$ -singularities of Y . Since $f|_S$ is crepant, $f(S)$ has only canonical singularity outside $\frac{1}{2}(1, 1, 1)$ -singularities of X except P . Since P is any $\frac{1}{2}(1, 1, 1)$ -singularity, we can find a member of $|-K_X|$ with only canonical singularities. □

8. SOME PROPERTIES OF \mathbb{Q} -FANO 3-FOLDS WITH ONLY $\frac{1}{2}(1, 1, 1)$ -SINGULARITIES AS ITS NON GORENSTEIN POINTS

Theorem 8.0. *Let X be a (not necessarily \mathbb{Q} -factorial) weak \mathbb{Q} -Fano 3-fold with $I(X) = 2$. Assume that X has the following properties:*

- (1) $|-K_X|$ has no fixed component. $|-K_X|$ has no base curve containing an index 2 point;
- (2) $|-K_X|$ has a member with only canonical singularities;
- (3) *there is no divisor contracted to a point by the morphism defined by $|-mK_X|$ for $m \gg 0$;*
- (4) $h^0(-K_X) \geq 4$;
- (5) *all non Gorenstein singularities of X are $\frac{1}{2}(1, 1, 1)$ -singularities.*

Then X can be deformed to a weak \mathbb{Q} -Fano 3-fold with only $\frac{1}{2}(1, 1, 1)$ -singularities as its singularities.

Proof. Let N be the number of $\frac{1}{2}(1, 1, 1)$ -singularities. We will prove this theorem by induction of N . We treat the case that $N = 0$ later. First we prove that if the assertion holds in case of $N - 1$, the assertion holds also in case of N (hence we assume that $N > 0$).

By assumption (4), $h^0(-K_X) \geq 4$ and hence by Riemann-Roch theorem and KKV vanishing theorem, we have $(-K_X)^3 \geq 2$. Let P be any $\frac{1}{2}(1, 1, 1)$ -singularity and $f : Y \rightarrow X$ the blow up at P . Then by Proposition 4.1, Y is a weak \mathbb{Q} -Fano 3-fold.

Then we verify that the assumption (1)~(5) hold for Y . (5) is clear. Since $f^{-1}|-K_X| = |-K_Y|$. (1), (2) and (4) follows. For (3), we assume that there is a divisor F on Y which is contracted to a point by the morphism defined by $|-mK_Y|$. If $E \cap F \neq \emptyset$, then $E \cap F$ is a curve since E is a Cartier divisor. By the nature of F , $E \cap F$ is numerically trivial for $-K_Y$ but by the nature of E , $E \cap F$ is numerically negative for $-K_Y$, a contradiction. Hence $E \cap F = \emptyset$. Then however $f(F)$ is contracted to a point by the morphism defined by $|-mK_X|$, a contradiction. Hence Y satisfies (1)~(5). By the assumption of the induction, the Gorenstein points of Y are smoothable. Let $\mathcal{Y} \rightarrow \Delta$ be a 1-parameter smoothing of Gorenstein points of Y . Then by [KoMo, Proposition 11.4], we obtain the deformation $\mathcal{X} \rightarrow \Delta$ of X which satisfies the commutative diagram

$$\begin{array}{ccc} \mathcal{Y} & \rightarrow & \mathcal{X} \\ \searrow & & \swarrow \\ & \Delta & \end{array}.$$

Then $\mathcal{Y}_t \rightarrow \mathcal{X}_t$ is an E_5 type contraction for $t \in \Delta$ since a contraction of type E_5 is stable under a deformation by [Kod2]. Hence X_t is a smoothing of Gorenstein points.

Next we prove the assertion in case $N = 0$. The proof is the same as one of [Na] except the following claim:

Claim. *Let D be a member of $|-K_X|$ with only canonical singularities. Then $\text{Pic}X \rightarrow \text{Pic}D$ is an injection.*

Proof. The proof is similar to one of [Na, Proposition 2] by virtue of the assumption (3).

□

□

Corollary 8.1. *Let X , Y , h and N as in the main theorem. Let $g : Y \rightarrow Z$ be the anti-canonical model. Assume that X has only $\frac{1}{2}(1, 1, 1)$ -singularities as its non Gorenstein points. Then if $N > 1$ (resp. $N = 1$), Z can be deformed to a \mathbb{Q} -Fano 3-fold Z' with $\rho(Z') = 1$ and $F(Z') = \frac{1}{2}$ which has only $N - 1$ $\frac{1}{2}(1, 1, 1)$ -singularities as its singularities and $h^0(-K_{Z'}) = h$ (resp. a smooth Fano 3-fold Z' with $\rho(Z') = 1$, $F(Z') = 1$ and $h^0(-K_{Z'}) = h$.)*

Proof. To apply Theorem 8.0, we only have to check that Z satisfies the assumption (1) and (5). By Proposition 7.0, any flopping curve does not contain a $\frac{1}{2}(1, 1, 1)$ -singularity. So the assumption (1) holds. Hence the assumption (5) is also satisfied by Proposition 7.0.

By these, we can apply Theorem 8.0 and Z can be deformed to a \mathbb{Q} -Fano 3-fold Z' with only $\frac{1}{2}(1, 1, 1)$ -singularities. We can prove by the proof of [KoMo, Corollary 12.3.4] that $\rho(Z') = 1$. If $N > 1$, $F(Z') = \frac{1}{2}$ by [San1] and [San2]. If $N = 1$, we have clearly $F(Z') = 1$. Hence we are done. \square

The next result is a first step for the classification of Mukai's type [Mu3, Theorem 1.10].

Theorem 8.2 (Embedding into a Weighted Projective Space). *Let X be a (not necessarily \mathbb{Q} -factorial) \mathbb{Q} -Fano 3-fold with canonical singularities and $I(X) = 2$. Assume that X has the following properties:*

- (1) $|-K_X|$ is indecomposable, i.e., $|-K_X|$ contains no member which is a sum of two movable Weil divisors;
- (2) $|-K_X|$ has no base curve containing an index 2 point;
- (3) $|-K_X|$ has a member with only canonical singularities;
- (4) for any index 2 point, there is a smooth rational curve l through it such that $-K_X.l = \frac{1}{2}$;
- (5) $h^0(X, \mathcal{O}(-K_X)) \geq 4$;
- (6) all non Gorenstein singularities of X are $\frac{1}{2}(1, 1, 1)$ -singularities.

Then except the following two cases (a) and (b), X is embedded into $\mathbb{P}(1^h, 2^N)$ and $-K_X$ is the restriction of $\mathcal{O}(1)$, where $h := h^0(-K_X)$ and N is the number of $\frac{1}{2}(1, 1, 1)$ -singularities:

- (a) $\Phi_{|-K_X|}$ is a double cover of \mathbb{P}^3 branched along a sextic.
- (b) $\Phi_{|-K_X|}$ is a double cover of a quadric hypersurface branched along the intersection with a quartic.

(Note that in case (a),

$$X \simeq ((6) \subset \mathbb{P}(1^4, 3)).$$

Note also that in case (b),

$$X \simeq ((2, 4) \subset \mathbb{P}(1^5, 2))$$

but the number of weight 2 is not equal to the number of non Gorenstein point.)

Furthermore X is an intersection of weighted hypersurfaces of degree ≤ 6 in $\mathbb{P}(1^h, 2^N)$.

If $h = 4$ and $N = 1$, then $X \simeq ((5) \subset \mathbb{P}(1^4, 2))$.

If $h = 4$ and $N = 2$, then $X \simeq ((3, 4) \subset \mathbb{P}(1^4, 2^2))$.

If $h = 5$ and $N = 1$, then $X \simeq ((3, 3) \subset \mathbb{P}(1^5, 2))$.

Proof. We prove this by induction of N .

In case $N = 0$, the assertion follows from [Mu3, Theorem 6.5 and Proposition 7.8].

Next we prove that if the assertion holds in case X has $N-1$ $\frac{1}{2}(1, 1, 1)$ -singularities, then so does it in case X has N $\frac{1}{2}(1, 1, 1)$ -singularities. Let X be a \mathbb{Q} -Fano 3-fold satisfying the assumptions of this theorem and with N $\frac{1}{2}(1, 1, 1)$ -singularities. Let $f : Y \rightarrow X$ be the blow up at a $\frac{1}{2}(1, 1, 1)$ -singularity. Let E be the exceptional divisor of f . Then Y is a weak \mathbb{Q} -Fano 3-fold by Proposition 4.1. By the assumption (5), Y is not \mathbb{Q} -Fano 3-fold. Let $g : Y \rightarrow Z$ be the morphism defined by a sufficient multiple of $-K_Y$ and $\overline{E} := \pi^*(E)$.

Claim 1. *Z satisfies the assumption of this theorem and has $N - 1$ $\frac{1}{2}(1, 1, 1)$ -singularities.*

Proof. By $-K_Y = g^*(-K_Z)$, if $|-K_Z|$ is decomposable, $|-K_X|$ must be decomposable, a contradiction. Hence (1) is satisfied.

By (2) for X , neither $|-K_Y|$ has a base curve containing an index 2 point. Hence any g -exceptional curve does not contain an index 2 point. So by $-K_Y = g^*(-K_Z)$, (2) is satisfied and (6) is also satisfied.

Let D be a member of $|-K_X|$ with only canonical singularities. Then the strict transform D' of D on Y has the same property since $D' \rightarrow D$ is crepant. Since $D' \rightarrow g(D')$ is crepant, $g(D')$ has also the same property. Hence (3) is satisfied.

By $-K_Y = g^*(-K_Z)$ and $h^0(-K_Y) = h^0(-K_X)$, we know that (5) is satisfied.

We show last that (4) is satisfied. If Z is Gorenstein, there is nothing to prove. If Z is non Gorenstein, let Q be any $\frac{1}{2}(1, 1, 1)$ -singularity and we will denote the corresponding points on Y and X also by Q . Then by (4) for X , there is a curve $Q \in l$ on X as stated in (4). For the strict transform l' of l on Y , we have $-K_Y.l' = \frac{1}{2}$ or 0. But if the latter case occurs, l' is a base curve of $|-K_Y|$ containing an index 2 point Q , a contradiction. Hence $-K_Y.l' = \frac{1}{2}$ and then $-K_Z.g(l') = \frac{1}{2}$. By blowing up Q , Z becomes a weak \mathbb{Q} -Fano 3-fold by Proposition 4.1 and (5) for Z . Then $g(l')$ become a curve contracted by an multi-anti-canonical morphism. Hence $g(l')$ is a smooth rational curve. Now we complete the proof of the claim. \square

Hence by the assumption of the induction, the following three cases occur:

Case α . $Z \subset \mathbb{P}(1^h, 2^{N-1})$ and $-K_Z = \mathcal{O}_Z(1)$;

Case β . Z is of type (a);

Case γ . Z is of type (b).

Claim 2. *$Bs|-K_X|$ coincides with $\frac{1}{2}(1, 1, 1)$ -singularities as a set.*

Proof. If $N = 0$, the assertion follows from [Mu3, Theorem 6.5 and Proposition 7.8]. Hence by Claim 1, the assertion follows by induction with respect to the number of $\frac{1}{2}(1, 1, 1)$ -singularities. \square

Case α . We first show that $\overline{E} \simeq E$. By Claim 2, the similar assertion holds for $|-K_Y|$. Hence $H^0(\mathcal{O}_Y(-K_Y)) \rightarrow H^0(\mathcal{O}_E(-K_Y)) \simeq H^0(\mathcal{O}_{\mathbb{P}^2}(1))$ is surjective. Hence $H^0(\mathcal{O}_Y(-mK_Y)) \rightarrow H^0(\mathcal{O}_E(-mK_Y))$ is also surjective for all $m \geq 0$ since $\bigoplus_{m \geq 0} H^0(\mathcal{O}_{\mathbb{P}^2}(m))$ is simply generated. So $\overline{E} \simeq E$ since g is defined by $-mK_Y$ for some $m > 0$.

We note here that there is an elementary transformation $\mathbb{P}(1^h, 2^N) \dashrightarrow \mathbb{P}(1^h, 2^{N-1})$ which is decomposed as follows:

Let \mathbb{P} be the projective bundle over $\mathbb{P}(1^h, 2^{N-1})$ whose vector bundle is $\mathcal{O} \oplus \mathcal{O}(-2)$ and T the effective tautological divisor (which is unique). Let a be the contraction morphism of T . Then $a(\mathbb{P})$ is isomorphic to $\mathbb{P}(1^h, 2^N)$. Let $b : \mathbb{P} \rightarrow \mathbb{P}(1^h, 2^{N-1})$ be the natural projection. Then our elementary transformation is $b \circ a^{-1}$.

We seek a natural morphism $Y \rightarrow \mathbb{P}$. For this, we prove that there is a natural surjection $g^*(\mathcal{O}_Z \oplus \mathcal{O}_Z(-2)) \rightarrow \mathcal{O}_Y(E)$.

There is the natural injection $\mathcal{O}_Y(-E) \rightarrow \mathcal{O}_Y$ which represents $\mathcal{O}_Y(-E)$ as the ideal sheaf of E . By Theorem 4.0, there is a member $S \in |-2K_X|$ such that $f^*S \cap E = \emptyset$. Associated to S , there is an injection $\mathcal{O}_Y(-f^*S) \rightarrow \mathcal{O}_Y$. This gives

an injection $\mathcal{O}_Y(-E) \rightarrow g^*\mathcal{O}_Z(2)$ since $g^*\mathcal{O}_Z(2) \simeq \mathcal{O}_Y(-2K_Y)$, $-f^*(-2K_X) \sim -(-2K_Y) - E$. By these, we can define a injection $\mathcal{O}_Y(-E) \rightarrow g^*(\mathcal{O}_Z \oplus \mathcal{O}_Z(2))$. Since $f^*S \cap E = \phi$, the cokernel of this map is locally free and hence the dual of this map is a surjection. Let $\iota : Y \rightarrow \mathbb{P}$ be the morphism over $\mathbb{P}(1^h, 2^{N-1})$ associated to the surjection $d : g^*(\mathcal{O}_Z \oplus \mathcal{O}_Z(-2)) \rightarrow \mathcal{O}_Y(E)$. By the definition of ι , we have $\iota(E) = T|_{\iota(Y)}$. In particular $\iota(E)$ is Cartier on $\iota(Y)$. Since $\overline{E} \simeq \mathbb{P}^2$, $\iota(E)$ is also \mathbb{P}^2 by the Zariski's Main Theorem. Hence $\iota(Y)$ is smooth at points of $\iota(E)$.

Claim 3. $\iota(Y)$ is normal.

Proof. It suffices to prove that $\iota_*\mathcal{O}_Y = \mathcal{O}_{\iota(Y)}$. The natural morphism $\mathcal{O}_{\iota(Y)} \rightarrow \iota_*\mathcal{O}_Y$ is injective since the kernel is at most torsion sheaf. Let \mathcal{C} be its cokernel. We will prove that $p_*\mathcal{C} = 0$. By the exact sequence

$$0 \rightarrow \mathcal{O}_{\iota(Y)} \rightarrow \iota_*\mathcal{O}_Y \rightarrow \mathcal{C} \rightarrow 0,$$

we have

$$0 \rightarrow p_*\mathcal{O}_{\iota(Y)} \rightarrow p_*\iota_*\mathcal{O}_Y \rightarrow p_*\mathcal{C} \rightarrow R^1p_*\mathcal{O}_{\iota(Y)}.$$

Since $p_*\mathcal{O}_{\iota(Y)} \rightarrow p_*\iota_*\mathcal{O}_Y$ is an isomorphism, it suffices to prove that $R^1p_*\mathcal{O}_{\iota(Y)} = 0$. Consider the exact sequence

$$0 \rightarrow \mathcal{I}_{\iota(Y)} \rightarrow \mathcal{O}_{\mathbb{P}} \rightarrow \mathcal{O}_{\iota(Y)} \rightarrow 0.$$

Since the dimension of a fiber of $p \leq 1$, we have $R^2p_*\mathcal{I}_{\iota(Y)} = 0$. Since \mathbb{P} is a \mathbb{P}^1 -bundle, we have $R^1p_*\mathcal{O}_{\mathbb{P}} = 0$. Thus we obtain $R^1p_*\mathcal{O}_{\iota(Y)} = 0$ and we are done.

Since every fiber of $g : Y \rightarrow Z$ intersects $\iota(E)$ and $\iota(Y)$ is smooth at points of $\iota(E)$, any fiber is not contained in the singular locus of $\iota(Y)$. Let l be any 1-dimensional fiber of g . By the theorem on formal functions, we have $\mathcal{C} \otimes \mathcal{O}_l = 0$ because $\dim \text{Supp } \mathcal{C} \otimes \mathcal{O}_l = 0$ (note that l is not contained in the singular locus of $\iota(Y)$) and $p_*\mathcal{C} = 0$. Hence by Nakayama's lemma, $\mathcal{C} = 0$. \square

Hence $\iota : Y \rightarrow \iota(Y)$ is finite and birational and $\iota(Y)$ is normal, it is an isomorphism by the Zariski's Main Theorem. Hence $X \simeq a(\iota(Y))$ is naturally embedded into $\mathbb{P}(1^h, 2^N)$ and $-K_X = \mathcal{O}(1)$.

Case β . Let $g' : Y \rightarrow Z \rightarrow \mathbb{P}^3$ be the composition of g and the double covering of \mathbb{P}^3 branched along a sextic. Consider the \mathbb{P}^1 -bundle $\mathbb{P}(\mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(-2))$ and denote it by \mathbb{P}' . Let $b' : \mathbb{P}' \rightarrow \mathbb{P}^3$ be the natural projection and T' the tautological divisor. Note that by $1 = (-K_Y)^2E = (g'^*\mathcal{O}(1))^2E = (\mathcal{O}(1))^2g'_*E$, we have $P := g'(E) \simeq E \simeq \mathbb{P}^2$. As in the treatment of Case α , we have a morphism $\iota' : Y \rightarrow \mathbb{P}'$ over \mathbb{P}^3 associated to the surjection $g'^*(\mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(-2)) \rightarrow \mathcal{O}_Y(E)$. By the definition of ι' , we have $\iota'^*(T'|_{\iota'(Y)}) = E$. Since $\deg g' = 2$, $\deg \iota' = 1$ or 2 . But we know that $\deg \iota' = 1$ by $\iota'^*(T'|_{\iota'(Y)}) = E$. Hence $\deg b'|_{\iota'(Y)} = 2$. So we can write $\iota'(Y) \sim 2T' + aL$, where $L := b'^*\mathcal{O}(1)$ and a is an integer. By $K_{\mathbb{P}'} \sim -2T' - 6L$, we have $K_{\iota'(Y)} \sim (a-6)L|_{\iota'(Y)}$. For a line l in $T'|_{\iota'(Y)}$, we have $K_{\iota'(Y)}.l = -1$. So $a = 5$, which in turn shows that $K_Y = \iota'^*K_{\iota'(Y)}$. Hence $\iota'(Y)$ is normal since it is Gorenstein and $Y \simeq \iota'(Y)$ by the Zariski's Main Theorem.

Contracting T' , \mathbb{P}' is transformed into $\mathbb{P}(1^4, 2)$ and Y is transformed into X . Hence we have the assertion.

Case γ . We will prove this case does not occur. Let $g' : Y \rightarrow R$ be the composition of g and the double covering $\Phi|_{-K_X}|$. Then $g'(E)$ is a plane in R by the same reason as in Case β . Hence we can assume that in $\mathbb{P}(1^5, 2)$ (note that $Z \subset (1^5, 2)$ in this case), \overline{E} is $(x_4 = x_5 = y = 0)$, where x_i 's ($i = 1, 2, 3, 4, 5$) are the coordinates of degree 1 and y is the coordinate of degree 2. So the weighted equation of degree 2 of Z is the form $ay + x_4l_1(x) + x_5l_2(x)$, which in turn shows that $Z \simeq ((4) \subset \mathbb{P}(1^4))$, a contradiction.

This complete the induction.

Finally we describe the graded ring of X .

First we note that $|-2K_X|$ is free since $-2K_X = \mathcal{O}_X(2)$. So we can take a smooth curve which is the intersection of general members of $|-K_X|$ and $|-2K_X|$. We fix such a curve and denote it by C and $L := -K_X|_C$. Note that L is a Cartier divisor such that $K_C = 2L$. Since $-K_X = \mathcal{O}(1)$, we may assume that $C \subset \mathbb{P}(1^{h-1}, 2^{N-1})$. It suffices to describe the graded ring of C . It is done by [RM4, Theorem 3.4]. Let $R(C, L) := \bigoplus_{m \geq 0} H^0(\mathcal{O}_C(mL))$. Let $X_N \subset \mathbb{P}(1^h)$ be the image of the restriction of the projection $\mathbb{P}(1^h, 2^N) \dashrightarrow \mathbb{P}(1^h)$. The rational map $X \dashrightarrow X_N$ is a composition of blow ups of $\frac{1}{2}(1, 1, 1)$ -singularities and crepant contractions in case $h \geq 5$ (resp. a composition of blow ups of $\frac{1}{2}(1, 1, 1)$ -singularities, crepant contractions and the double covering of \mathbb{P}^3 in case $h = 4$). So the restriction of the projection to C is a birational map in case $h \geq 5$ (resp. a birational map or a double cover of a plane curve of degree ≥ 3 in case $h = 4$), which in turn show that the image of C by the morphism of L is not a normal rational curve in $\mathbb{P}(1^{h-1})$. Hence $H^0(\mathcal{O}_C(L)) \otimes H^0(\mathcal{O}_C(2L)) \rightarrow H^0(\mathcal{O}_C(3L))$ is surjective. (Note that $K_C = 2L$.) So by [RM4, Theorem 3.4], $R(C, L)$ is generated by elements of degree ≤ 2 and related by elements of degree ≤ 6 , which in turn show that the same things hold for $\bigoplus_{m \geq 0} H^0(\mathcal{O}_X(-mK_X))$. Let N' be the number of sub bases of degree 2 which do not come from degree 1. Since the above embedding $X \subset \mathbb{P}(1^h, 2^N)$ come from (possibly) some projection $(1^h, 2^{N'}) \dashrightarrow \mathbb{P}(1^h, 2^N)$, X is an intersection of weighted hypersurfaces of degree ≤ 6 .

Finally we determine X in 3 cases as in the statement of this theorem. It suffices to determine C as above. If $h = 4$ and $N = 1$, the assertion is clear. Assume that $h = 4$ and $N = 2$. If there is a relation of degree 2 in $R(C, L)$, the image of the restriction to C of the projection $\mathbb{P}(1^4, 2) \dashrightarrow \mathbb{P}(1^4)$ is a conic in \mathbb{P}^2 , a contradiction. Hence there is no relation of degree 2 in $R(C, L)$. Then we find easily the relation of $R(C, L)$.

Assume that $h = 5$ and $N = 1$. Note that we know that $C \subset \mathbb{P}^3$ and $\deg C = 9$. Hence if there is a relation of degree 2 in $R(C, L)$, there is exactly one relation and one degree 2 base which does not come from degree 1. But from this, we can see that there is 2 relation in degree 3, a contradiction to that $\deg C = 9$. Hence there is no relation of degree 2 in $R(C, L)$. The rest are easy calculations. Now we complete the proof of the main theorem. \square

Remark. The assumption that $h^0(-K_X) \geq 4$ is necessary for Theorem 8.2 by the existence of the following:

$$(X \simeq (((12) \subset \mathbb{P}(1^3, 4, 6))))$$

which satisfies $h^0(-K_X) = 3$.

Corollary 8.3. *Let X be a \mathbb{Q} -Fano 3-fold as in the main theorem. Assume that X has only $\frac{1}{2}(1, 1, 1)$ -singularities as its non Gorenstein points. Then X is embedded into $\mathbb{P}(1^h, 2^N)$ and $-K_X$ is the restriction of $\mathcal{O}(1)$, where $h := h^0(-K_X)$ and N is the number of $\frac{1}{2}(1, 1, 1)$ -singularities. Furthermore X is an intersection of weighted hypersurfaces of degree ≤ 6 .*

If $h = 4$ and $N = 1$, then $X \simeq ((5) \subset \mathbb{P}(1^4, 2))$.

If $h = 4$ and $N = 2$, then $X \simeq ((3, 4) \subset \mathbb{P}(1^4, 2^2))$.

If $h = 5$ and $N = 1$, then $X \simeq ((3, 3) \subset \mathbb{P}(1^5, 2))$.

Proof. By Corollary 6.3, Proposition 7.0 and Corollary 7.2, we can see that the assumptions of Theorem 8.2 are satisfied for X . Hence we are done. \square

By this Corollary 8.3, we can improve Theorem 4.0 for X as in Corollary 8.3 and Proposition 7.0 as follows:

Corollary 8.4. *Let X be a \mathbb{Q} -Fano 3-fold as in the main theorem. Then*

- (1) $-2K_X$ is very ample;
- (2) $|-K_X|$ is free outside $\frac{1}{2}(1, 1, 1)$ -singularities and its general member has only ordinary double points as its singularities.

Proof. The proof is clear from Corollary 8.3. \square

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